دفتر :

تفاضل و تكامل 3

calculus 3

إعـــداد
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اللجنة الأكاديمية لقسم الهندسة الصناعية

2025

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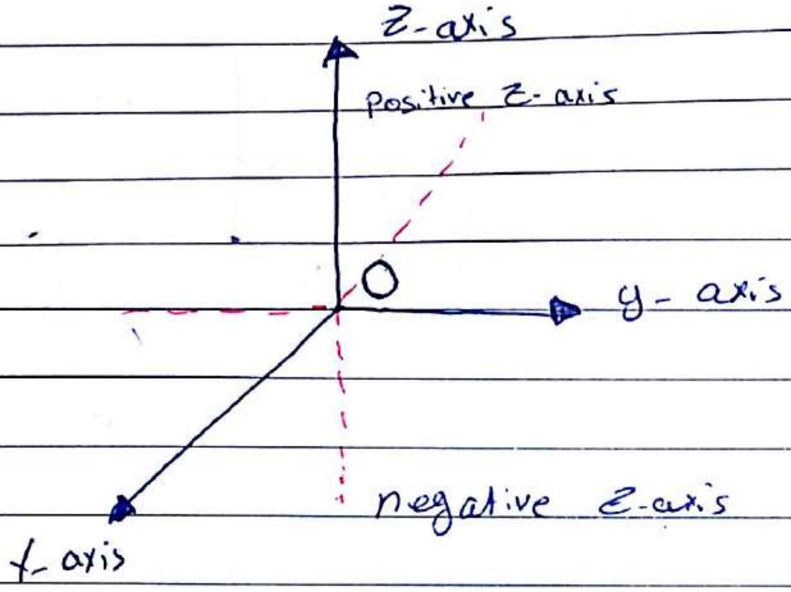
Ch.12 :> Vectors and Geometry of space

Sec. 12.1 The Three dimensional Coordinate systems

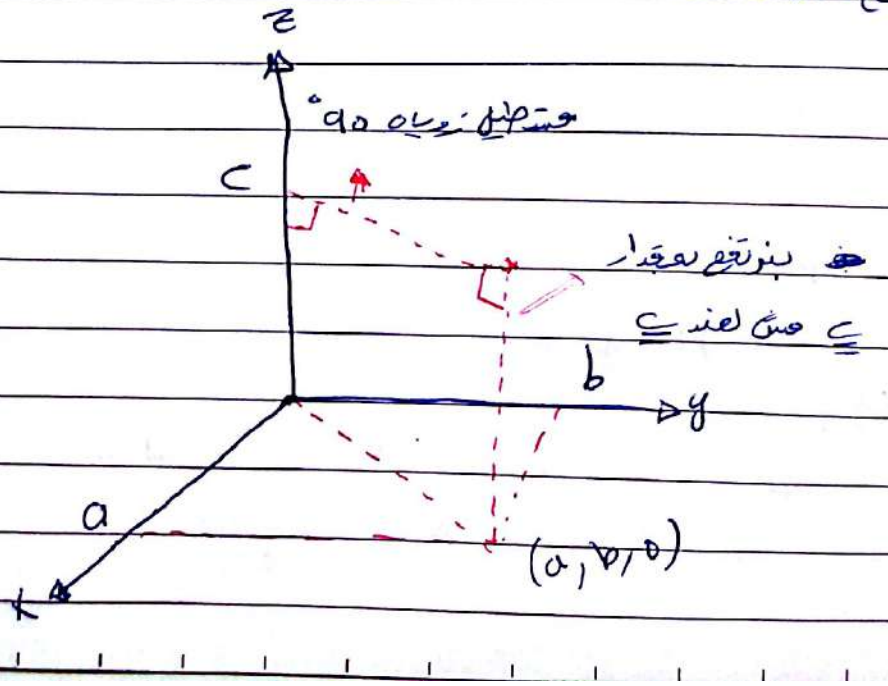
Suppose we have 3 perpendicular lines in the space that are intersected at the same point (pt.)



axes
⇓
axis . جمع

These axes called the Coordinate axes
The pt. of intersection is called the origin
A representation of a pt. $A(a,b,c)$ in the space as:

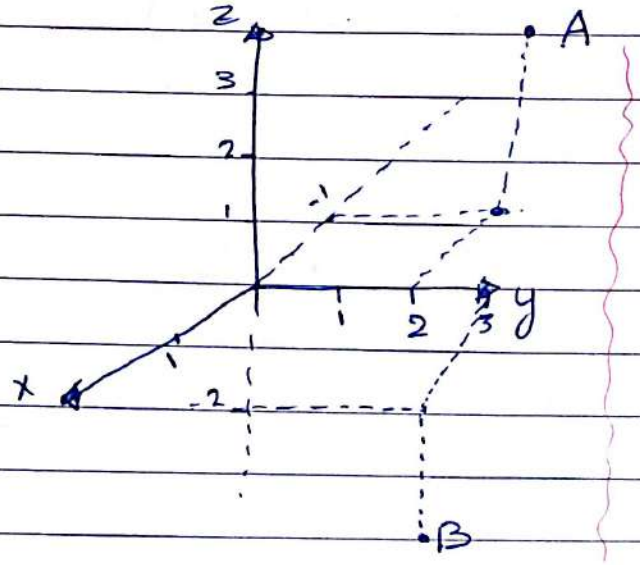
تمثيل

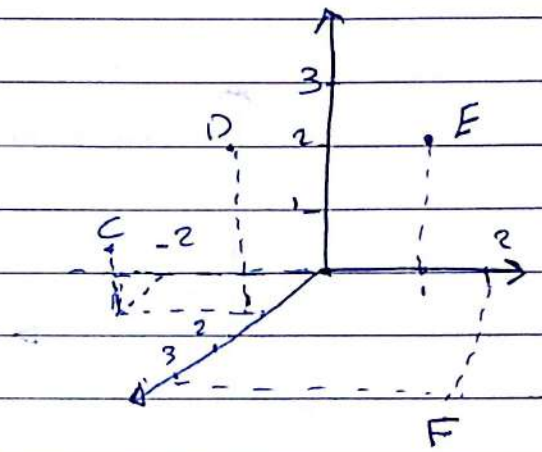


Ex: plot the graph of the pts. A (-1, 2, 3)   B (1, 3, -2)   C (1, -2, 1)

D (1, 0, 3)   E (0, 1, 2)   F (3, 2, 0)

G (1, 0, 0)   H (0, 0, -2)





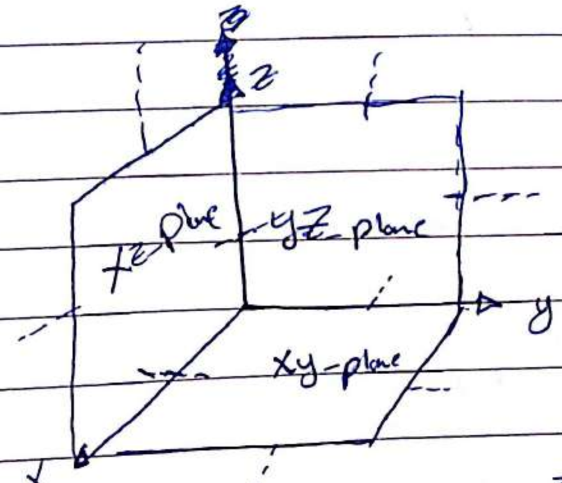
(a, b, c) : is a pt. in the space.

a: x- coordinate   # b : y- coordinate   # c : z- coordinate

المحاور الاحداثية تقع المستويات الاحداثية

* The coordinate planes are :-

xy- plane , xz - plane , yz- plane



المستويات الاحداثية تقسم الفضاء الى ٨ مستويات

These Coordinate Planes divide the space into 8 parts each part is called an octant

The first octant is (1^{st}) the octant that includes the positive coordinate axes

#: Remark: The graph of the equation (eq.):

1) $f(x,y)=0$ in the plane is a curve

2) $f(x,y,z)=0$ in the space is a surface

Ex: 1) $y=x^2$ is a curve in the ~~space~~ plane

2) $y=x^2$ is a surface in the space

× لو حاجة كانت في الـ plane ظهرت تطلع مجسمة space

Ex: sketch the graph of the following eqs. in the space

① $y=x^2$ ② $z=-y$ ③ $x=2$ ④ $x=0$

⑤ $y=0$ ⑥ $z=0$

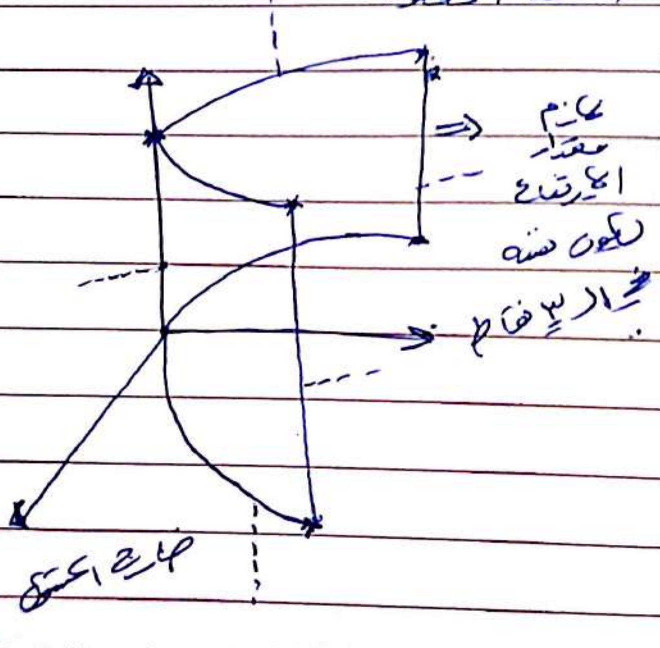
Sol:

①

١- المستوى المختار بالخطين الظاهرة ونرسم الثالث عادي

٢- نأخذ ٣ نقاط بشكل موازي للمحور الثالث

٣- نوصل النقاط بنفس المنحنية.



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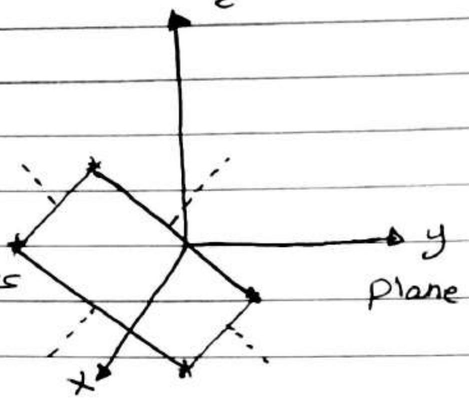
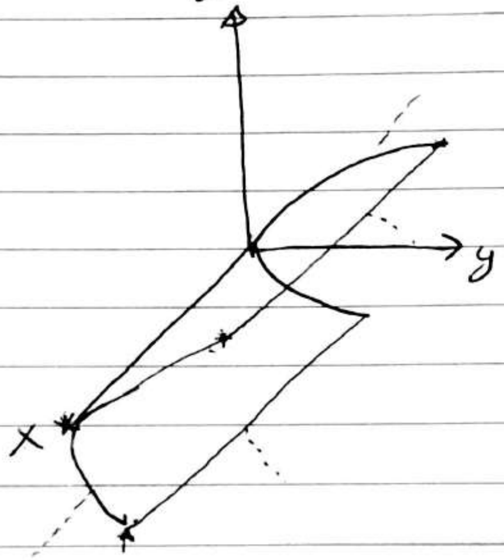
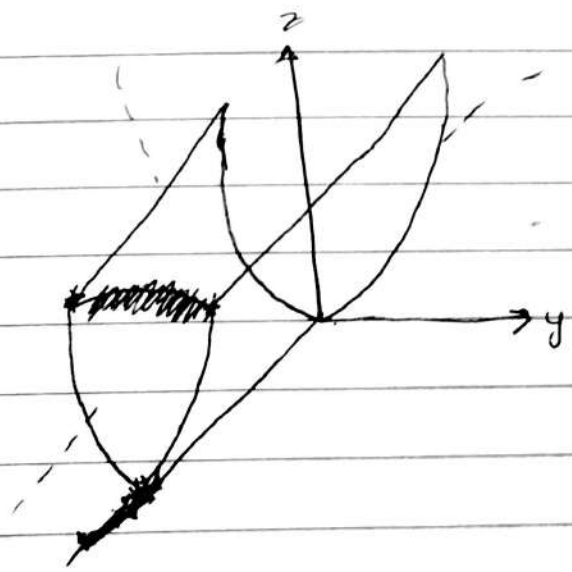


3 $y = z^2$

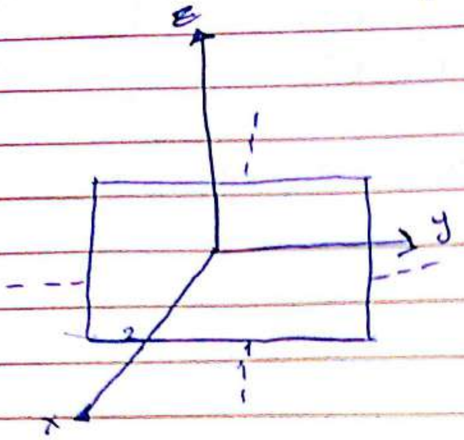


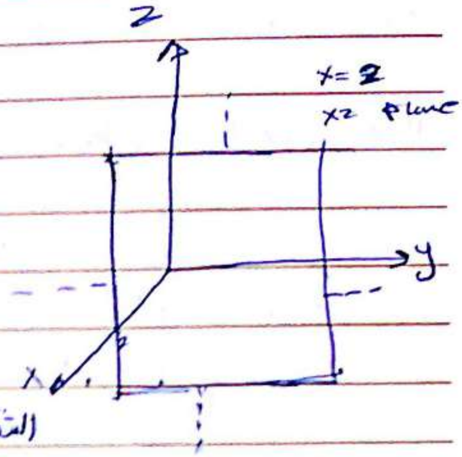
4 $z = y^2$



5. $x = 2 \Rightarrow x + 0z = 2$ أو $x + 0y = 2$

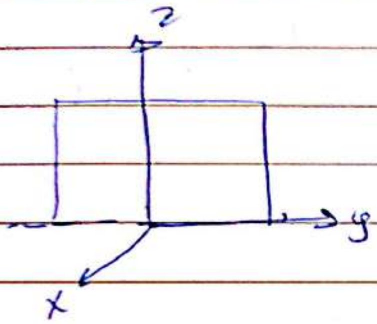
$x = 2 \Rightarrow$ eq. in x, y in the plane xy..





الشكلين نفسهم

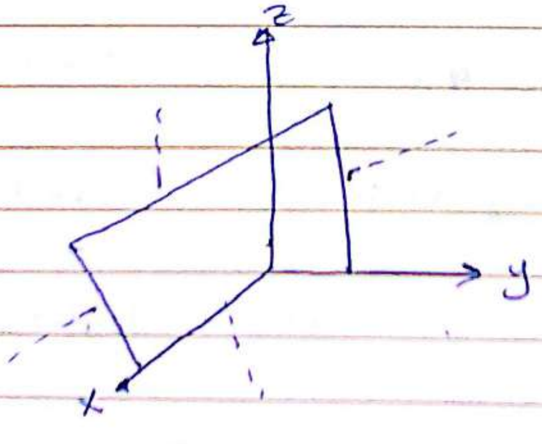
6. $x = 0$



zy-plane

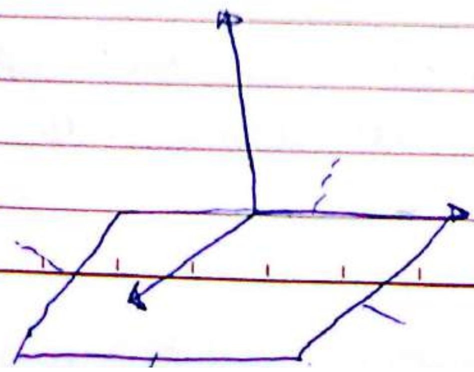
حطينا الصفر في معادلة

7. $y = 0$



xz-plane

8- $z = 0$



xy-plane

Definition (Def.) نصف قطر

The eq. of the sphere of radies r and center (a,b,c)
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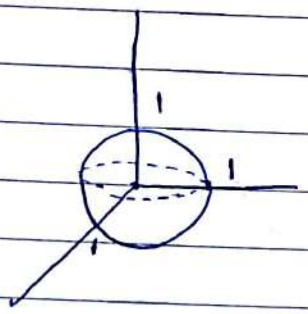
is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

The unit sphere is : $x^2+y^2+z^2=1$



ex: ① The eq. $x^2+(y+3)^2+(z-4)^2=9$ is an eq. of asphere of radios 3, center $(0,-3,4)$

② Show that the eq.
$2x^2+2y^2+2z^2+8x-12y+\cancel{\ }+12=0$
is a sphere and find its center, radies.

Sol: eq $\div 2 \Rightarrow x^2+4x+y^2-6y+z^2=-6$

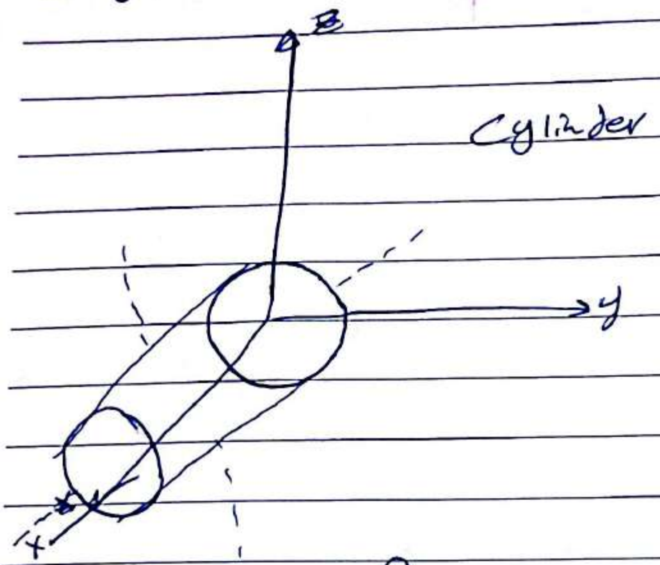
$(x^2+4x+4)+(y^2-6y+9)+z^2=-6+4+9$

$(x+2)^2+(y-3)^2+z^2=7 \Rightarrow$ is asphere

center $(-2,3,0)$ radies $=\sqrt{7}$

③ Sketch the graph of in the space
① $x^2+y^2=4$ ③ $x^2+z^2=16$
② $y^2+z^2=9$

2 $y^2 + z^2 = 9$



Notation: $\mathbb{R} = (-\infty, \infty)$

② $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x, y) : x, y \in \mathbb{R} \} = xy$- plane

③ $\mathbb{R}^3 = \{ (x, y, z) : x, y, z \in \mathbb{R} \}$

$=$ xyz- ~~plane~~ space = space

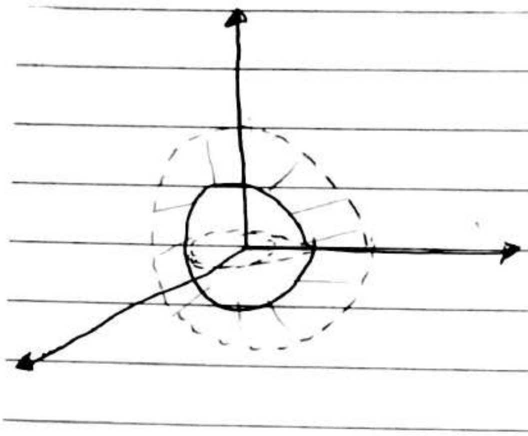
Examples: Sketch the regin in $\mathbb{R}^3$ that represent the inequalities:

1. $y^2 \leq z \leq y^2 + 1$
2. $1 \leq x^2 + y^2 + z^2 < 4$
3. $x^2 + y^2 + z^2 > 6z$
4. $x \geq 3$
5. $z \leq 0$

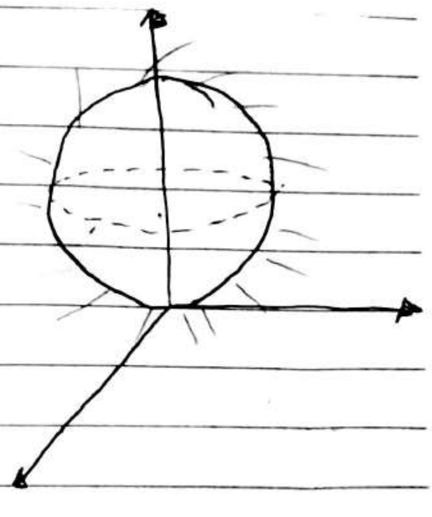
Soln.

[2] $x^2+y^2+z^2=1$

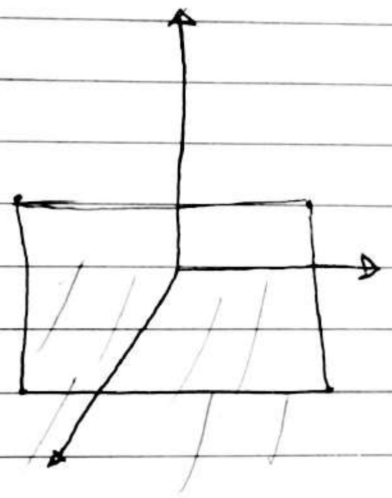
$x^2+y^2+z^2=4$



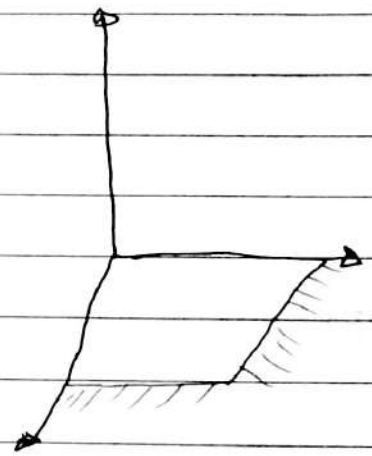
[3] $x^2+y^2+(z-3)^2=9$



[4] $x=3$



[5] $z=0$



Deff. The distance between two points $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$ is :-

$$\text{dist}(A,B) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Deff the mid point of the line segement joining two pts $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ is :-

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example: Find the eq. of the sphere if one of the diameter has end pt. $P(2,1,4)$, $Q(4,3,7)$

Sol. The center = mid pt. $= (3, 2, \frac{11}{2})$

$r = \frac{1}{2}\,\text{dist}(P,Q) = \frac{\sqrt{17}}{2}$

The eq. is $(x-3)^2 + (y-2)^2 + (z - \frac{11}{2})^2 = \frac{17}{4}$

Remark: The distance from a pt. $A(a,b,c)$ and the:

1) xy-plane is $|c|$
2) zy-plane is $|a|$
3) xz-plane is $|b|$

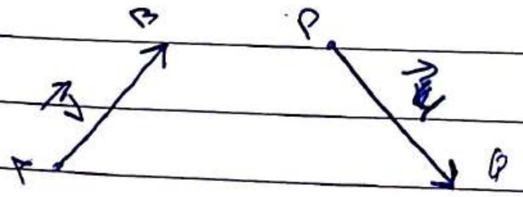
Example: Find the eq. of the sphere centered at $A(2,-1,-3)$ and tuches the xz-plane

Soln. $r = |-1| = 1$

$\Rightarrow (x-2)^2 + (y+1)^2 + (z+3)^2 = 1$

Sec 12.2 ... Vectors

Def: A vector $\vec{v}$ is a quantity that has:
1) Magnitude (or length) $|\vec{v}|$
2) Direction



* We can represent vectors using their initial and terminal pts

$\vec{V} = \vec{AB}$

$\vec{u} = \vec{PQ}$

* If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ then :-

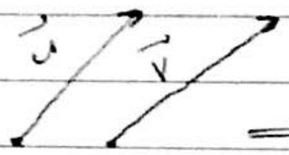
$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

* Def :- The Zero Vector $\vec{0}$ is a vector with the same initial and terminal pts.

$\vec{0} = \vec{AA} = \vec{BB}$

$\vec{0}$ vector of length 0 and in any direction

* Def :- The two vectors $\vec{u}, \vec{V}$ are equal $\vec{u} = \vec{V} \iff$ They have the same length and direction



$\Rightarrow \vec{u} = \vec{V}$

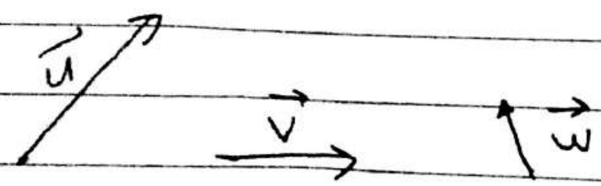
* Def :- let $\vec{u}$ be a vector, a is a number, $a\vec{u}$ is a vector with length

$|a\vec{u}| = |a| \cdot |\vec{u}|$ and it's :-

① In the Same direction of $\vec{u}$, if $a > 0$

② in the opposite direction of $\vec{u}$ if $a < 0$

Example :-



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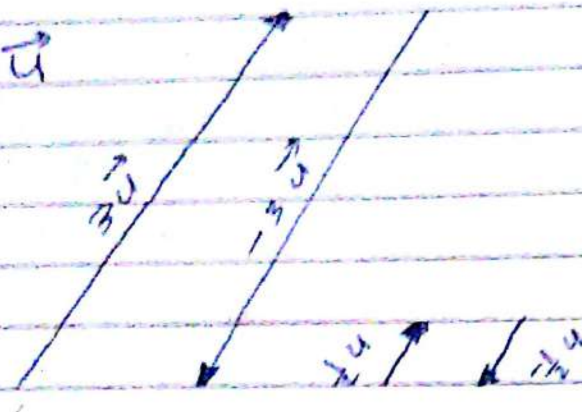
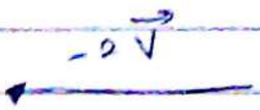
$\vec{u}$



$\vec{v}$

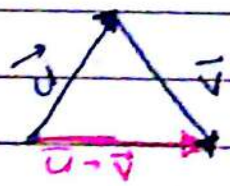


Remark

(1) $\vec{u}$, $-\vec{u}$ are of the same length but with opposite dir.

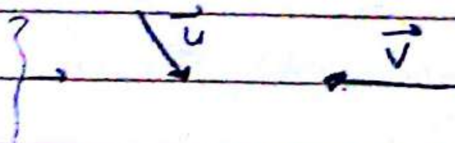
(2) $0\vec{u} = \vec{0}$

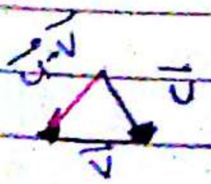
Def let $\vec{u}, \vec{v}$ be two vectors as in the figure, then $\vec{u}+\vec{v}$ is the vectors with initial pt as that of $\vec{u}$, and terminal pt. as that of $\vec{v}$.

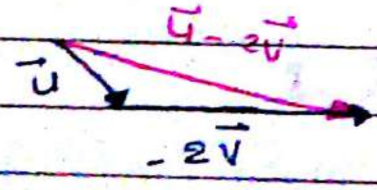


Ex: Sketch: $\vec{u}+\vec{v}$

: $\vec{u}+-2\vec{v}$







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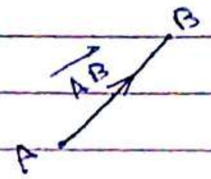
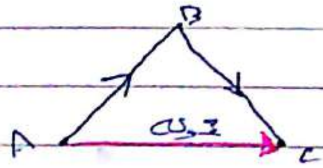
Remark :

$-\overrightarrow{AB} = \overrightarrow{BA}$



Remark :

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



Example :

let A, B, C be 3 pts.

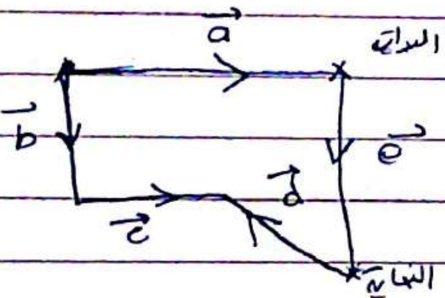
what is the vector :

$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC}$?!

Sol. $\underbrace{\overrightarrow{AB} + \overrightarrow{BC}}_{\overrightarrow{AC}} - \overrightarrow{AC} = \overrightarrow{AC} + -\overrightarrow{AC} = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$

Example :

write the vector $\vec{e}$ as a sum of the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$

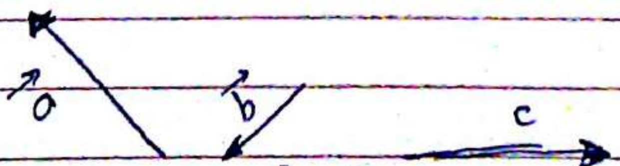


$\vec{e} = -\vec{a} + \vec{b} + \vec{c} + -\vec{d}$

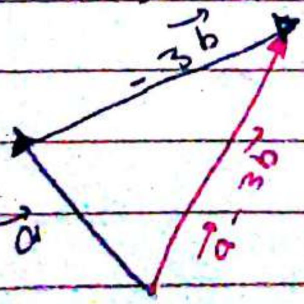
Example :

Draw the vectors [1] $\vec{a} - 3\vec{b}$ [2] $2\vec{a} - 2\vec{b} + \vec{c}$

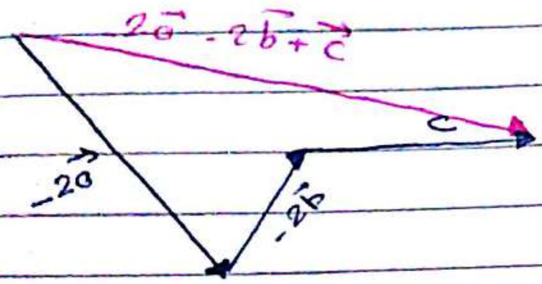
where



[1]



[2]



The Component form of a vector $\vec{V}$ in $\mathbb{R}^3$ is $\vec{V} = \langle a, b, c \rangle$

it's initial pt. is $O(0,0,0)$ and terminal pt. $P(a,b,c)$

$|V| = \sqrt{a^2+b^2+c^2}$

let $\vec{V} = \overrightarrow{AB}$ where $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

$\vec{v} = \langle x_2 - x_1 , y_2 - y_1 , z_2 - z_1 \rangle$

$\boxed{B - A}$ النهاية - البداية.

Example: let $\vec{V} = \langle 2, -1, 0 \rangle$

[1] $\vec{V} = \overrightarrow{OP}$, $O(0,0,0)$, $P = (2,-1,0)$

[2] $\vec{V} = \overrightarrow{AB}$, $A(2,4,6)$, $B = (4,3,5)$

[3] $\vec{V} = \overrightarrow{CD}$, $C(-3,0,-1)$, $D = (-1,-1,-1)$

properties: let $\vec{u}, \vec{V}, \vec{w}$ be Vectors in $\mathbb{R}^3$

[1] $\vec{V} = \vec{0} \iff \vec{V} = \langle 0, 0, 0 \rangle$

[2] let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, k: number

Then $k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

$\vec{u} + \vec{v} = \langle u_1 + v_1 , u_2 + v_2 , u_3 + v_3 \rangle$

$\boxed{3}$ $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

Example: let $\vec{u} = \langle -1, 2, 3 \rangle$, $\vec{v} = \langle 2, 1, 5 \rangle$
Find $|3\vec{u} - 2\vec{v}|$

Sol. $3\vec{u} - 2\vec{v} = \langle 3(-1) - 2(2) , 3(2) - 2(1) , 3(3) - 2(5) \rangle$

$= \langle -7, 4, -1 \rangle$

$|3\vec{u} - 2\vec{v}| = \sqrt{49 + 16 + 1} = \sqrt{66}$

Def: A unit vector denoted By $\hat{a}, \hat{b}, \hat{c}, \dots, \hat{u}, \hat{v}$
is a vector of length 1 $|\hat{a}| = 1$

Example → determine If the Vectors bellow are unit Vector or not e!

$\boxed{1}$ $\vec{u} = \langle \frac{1}{2} , \frac{1}{2} , \frac{\sqrt{3}}{2} \rangle$

$|\vec{u}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{5}{4}} \neq 1$ not unit vector

$\boxed{2}$ $\vec{w} = \langle \frac{-3}{4} , \frac{1}{2} , \frac{\sqrt{3}}{4} \rangle$

$|\vec{w}| = \sqrt{\frac{9}{16} + \frac{1}{4} + \frac{3}{16}} = \sqrt{1} = 1$

$\therefore$ $\vec{w}$ is a unit vector

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Exercise :

Find a s.t

$\vec{b} = \langle \frac{1}{5}, a, \frac{-1}{5} \rangle$ unit vector

Def : let $\vec{a} \neq 0$ then

$\frac{\vec{a}}{|\vec{a}|}$, $\frac{-\vec{a}}{|\vec{a}|}$ unit vectors

$\therefore \frac{\vec{a}}{|\vec{a}|}$ unit vector in the same direction of $\vec{a}$

$\therefore \frac{-\vec{a}}{|\vec{a}|}$ unit vector in the oppisite direction of $\vec{a}$

$k \frac{\vec{a}}{|\vec{a}|}$ vector of length $|k|$: in the same direction of $\vec{a}$ if $k > 0$

in the opposite direction of $\vec{a}$ if $k < 0$

Example : let $\vec{a} = \langle 2, -1, 3 \rangle$

1) A unit vector in :-

* the same direction of $\vec{a}$ is $\frac{\vec{a}}{|\vec{a}|} = \langle \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$

* " opposite " " " is $\frac{-\vec{a}}{|\vec{a}|} = \langle \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \rangle$

(2) A vector of length 0.1 in

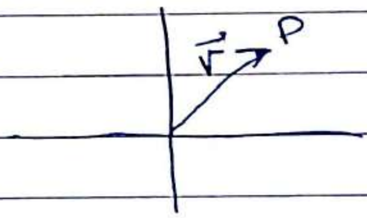
* The same direction of $\vec{a}$ is $0.1 \dfrac{\vec{a}}{|\vec{a}|} = \left\langle \dfrac{0.2}{\sqrt{14}}, \dfrac{-0.1}{\sqrt{14}}, \dfrac{0.3}{\sqrt{14}} \right\rangle$

* " opposite " " " is $-0.1 \dfrac{\vec{a}}{|\vec{a}|} = \left\langle \dfrac{-0.2}{\sqrt{14}}, \dfrac{0.1}{\sqrt{14}}, \dfrac{-0.3}{\sqrt{14}} \right\rangle$

Remark :

in $\mathbb{R}^2$, $\vec{v} = \langle a,b \rangle = \vec{OP}$, O $(0,0)$, $P = (a,b)$

$|\vec{v}| = \sqrt{a^2+b^2}$

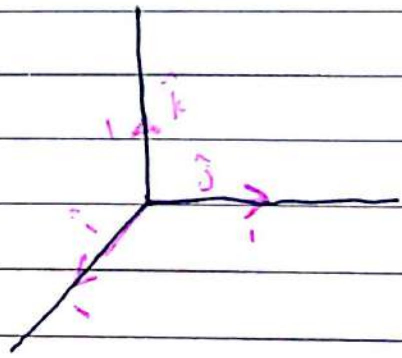


Remark : [1] In $\mathbb{R}^3$ the basis unit vectors

$\hat{i} = \langle 1,0,0 \rangle$

$\hat{j} = \langle 0,1,0 \rangle$

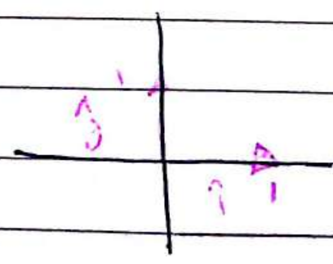
$\hat{k} = \langle 0,0,1 \rangle$



[2] In $\mathbb{R}^2$ basis vectors

$\hat{i} = \langle 1,0 \rangle$

$\hat{j} = \langle 0,1 \rangle$



$\vec{v} = \langle a, b, c \rangle$

$= \langle a,0,0\rangle + \langle 0,b,0\rangle + \langle 0,0,c\rangle$

$= a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$

$= a\hat{i} + b\hat{j} + c\hat{k}$

Example: let $\vec{a} = 2\hat{i} - 3\hat{j}$

$\vec{b} = \langle 2, 4, -3 \rangle$

$\vec{a} - \vec{b} = (2-2)\hat{i} + (-3-4)\hat{j} + (0 - -3)k$

$= -7\hat{j} + 3\hat{k}$

$|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$

Notation:

V_2 : the set of All vectors in $\mathbb{R}^2$

V_3 : " " " " " $\mathbb{R}^3$

See 12.3 Dot product.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

The dot product of $\vec{a}$ and $\vec{b}$ is :-

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\vec{a}, \vec{b} \in \mathbb{R}$

Example let $\vec{a} = 2\hat{i} - 3\hat{j}$

$\vec{b} = \langle 5, 7, -3 \rangle$

$\vec{a} \cdot \vec{b} = \langle 2, -3, 0 \rangle , \langle 5, 7, -3 \rangle$

$= (2)5 + (3)(7) + (0)(-3)$

$= -11$

properties:

$\boxed{1}$ $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\boxed{2}$ $\vec{a} \cdot (\vec{b} \pm \vec{c}) = \vec{a} \cdot \vec{b} \pm \vec{a} \cdot \vec{c}$

$\boxed{3}$ $(\vec{a}+\vec{b}) \cdot (\vec{c}+\vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$

$\boxed{4}$ $(a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$

Remark: $\vec{a} = \langle a_1 , a_2 , a_3 \rangle$

$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$

$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$

$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$

Rule: let $\vec{u}, \vec{v}$ be vectors a, b be scalars

then:

$\boxed{1}$ $|a\vec{u} + b\vec{v}|^2 = a^2 |\vec{u}|^2 + 2ab\, \vec{u} \cdot \vec{v} + b^2 |\vec{v}|^2$

$\boxed{2}$ $|a\vec{u} - b\vec{v}|^2 = a^2 |\vec{u}|^2 - 2ab\, \vec{u} \cdot \vec{v} + b^2 |\vec{v}|^2$

Proof:

$\boxed{2}$ $|a\vec{u} - b\vec{v}|^2 = (a\vec{u} - b\vec{v}) \cdot (a\vec{u} - b\vec{v})$

$= a\vec{u} \cdot a\vec{u} - a\vec{u} \cdot b\vec{v} - b\vec{v} \cdot a\vec{u} + b\vec{v} \cdot b\vec{v}$

$= a^2\, \vec{u} \cdot \vec{u} - ab\, \vec{u} \cdot \vec{v} - ab\, \vec{v} \cdot \vec{u} + b^2\, \vec{v} \cdot \vec{v}$

$= a^2 |\vec{u}|^2 - 2ab\, \vec{u} \cdot \vec{v} + b^2 |\vec{v}|^2$

Scanned by CamScanner

Ex :> let $|\vec{a}| = 3$, $|\vec{b}| = 6$

and $|2\vec{a} - 3\vec{b}| = 12$

[1] find $\vec{a}\cdot\vec{b}$ $\boxed{2}$ find $|\vec{a} + 4\vec{b}|$

Soln. $|2\vec{a} - 3\vec{b}|^2 = (12)^2$

$4|\vec{a}|^2 - (2)(2)(3)\ \vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 144$

$36 - 12\ \vec{a}\cdot\vec{b} + 324 = 144$

$-12\ \vec{a}\cdot\vec{b} = -216$

$\vec{a}\cdot\vec{b} = \frac{216}{12}$

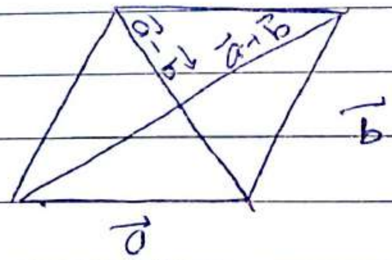
$\boxed{2}$ $|\vec{a}\cdot 4\vec{b}|^2 = |\vec{a}|^2 + 2(4)\ \vec{a}\cdot\vec{b} + 16|\vec{b}|^2$

$= \sqrt{}$ لازم تكمل الفرع الأول عشان تقدر تعمل الفرع التاني.

قانون متوازي الاضلاع

H.~~Role~~ Exercise :- (parallelegram low)

$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$



Pf.

$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 + |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |b|^2$

Example :- If $|\vec{a}| = 2$, $|\vec{b}| = 3$, $|5\vec{a} - 4\vec{b}| = 4$

Find $|5\vec{a} + 4\vec{b}|$?

Five Apple

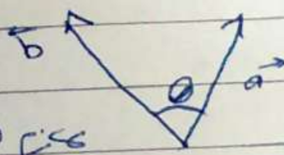
Soln. $|5\vec{a}+4\vec{b}|^2 + |5\vec{a}-4\vec{b}|^2 = 2\left(|5\vec{a}|^2 + |4\vec{b}|^2\right)$

$|5\vec{a}+4\vec{b}|^2 + 16 = 2\left(25(4) + 16(9)\right)$

Rule : $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

where is the angle between $\vec{a}, \vec{b}$ $\theta \in [0, \pi]$

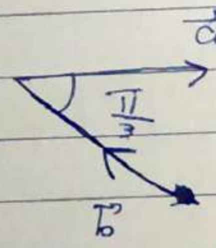
as in the figure



عشان نقطة البداية نفسها ⟸ نجيب الزاوية المحصورة 180

Example : find $\vec{a}\cdot\vec{b}$

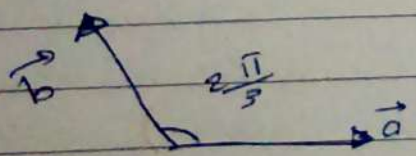
where $|\vec{a}| = 2|\vec{b}| = 6$



نقطة البداية مش نفسها

$|\vec{a}| = 6$

$2|\vec{b}| = 6 \Rightarrow |\vec{b}| = 3$



$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = (6)(3)\cos\left(\frac{2\pi}{3}\right)$

$= 18\left(-\cos\frac{\pi}{3}\right)$

$= -18\left(\frac{1}{2}\right) = -9$

	$\pi/6$	$\pi/3$	$\pi/3$
Sin	$1/2$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
Cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$

Remark : let θ be the angle between $\vec{a}, \vec{b}$

1) $\vec{a}\cdot\vec{b} > 0 \Leftrightarrow \theta$ acute angle حادة

2) $\vec{a}\cdot\vec{b} < 0 \Leftrightarrow \theta$ obtuse angle منفرجة

[3] $\vec{a}\cdot\vec{b} = 0 \iff \theta = \frac{\pi}{2}$

$\iff \vec{a}, \vec{b}$ are Perpendicular
orthogonal
normal

Example Are $\vec{a}, \vec{b}$ perpendicular ?!

[1] $\vec{a} = 3\hat{i} - 4\hat{j} + \hat{k}$
$\vec{b} = \hat{i} - 3\hat{k}$

$\boxed{\text{yes since } \vec{a}\cdot\vec{b} = (3)(1) + -4(0) + (1)(-3) = 0}$

[2] $\vec{a} = 2\hat{i} - 5\hat{k}$
$\vec{b} = \langle 1, 1, 1 \rangle$

$\boxed{\text{No since } \vec{a}\cdot\vec{b} = (2)(1) + (0)(1) + -5(1) \neq 0}$

Note : $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ where $\vec{a}, \vec{b}$ are two Vectors

$\theta = \cos^{-1} \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

Example Find the angle between
$\vec{u} = 3\hat{j} - 2\hat{k}$
$\vec{v} = 3\hat{i} + 6\hat{k}$

$\theta = \cos^{-1}\left(\frac{-12}{\sqrt{13} \times \sqrt{45}}\right)$

Exercies: If $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$ are orthogonal then show that $|\vec{u}| = |\vec{v}|$

∴ pf: $(\vec{u}+\vec{v}) \cdot (\vec{u}-\vec{v}) = 0$

$|\vec{u}|^2 - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - |\vec{v}|^2 = 0$

$|u|^2 = |\vec{v}|^2 \Longrightarrow |u| = |v|$

∴ Recall that: $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, $\theta \in [0, \pi]$

∴ Def: The direction angles of a vector $\vec{a}$, are: α, β, γ

where α is angle between $\vec{a}$ and $\hat{i}$

β is " " " " $\hat{j}$

γ is " " " " $\hat{k}$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$, are direction cosines of $\vec{a}$

Rule: let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

∴ $\cos\alpha = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}|\,|\hat{i}|} = \dfrac{a_1}{|\vec{a}|} \Longrightarrow \boxed{\alpha = \cos^{-1}\dfrac{a_1}{|\vec{a}|}}$

∴ $\cos\beta = \dfrac{\vec{a}\cdot\hat{j}}{|\vec{a}|\,|\hat{j}|} = \dfrac{a_2}{|\vec{a}|} \Longrightarrow \boxed{\beta = \cos^{-1}\dfrac{a_2}{|\vec{a}|}}$

∴ $\cos\gamma = \dfrac{\vec{a}\cdot\hat{k}}{|\vec{a}|\,|\hat{k}|} = \dfrac{a_3}{|\vec{a}|} \Longrightarrow \boxed{\gamma = \cos^{-1}\dfrac{a_3}{|\vec{a}|}}$

Rule : $\vec{a} = \langle |\vec{a}| \cos\alpha , |\vec{a}| \cos\beta , |\vec{a}| \cos\gamma \rangle$

pf : $\vec{a} = \langle |\vec{a}| \cos\alpha , |\vec{a}| \cos\beta , |\vec{a}| \cos\gamma \rangle$

$|\vec{a}|^2 = |\vec{a}|^2 \cos^2\alpha + |\vec{a}|^2 \cos^2\beta + |\vec{a}|^2 \cos^2\gamma \quad \div |\vec{a}|^2$

$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

Example : find the direction cosines and direction angles of $\vec{a} = \langle 1 , -2 , \sqrt{3} \rangle$

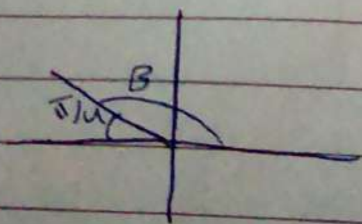
Sol : $\cos\alpha = \dfrac{1}{2\sqrt{2}}$

$\cos\beta = \dfrac{-2}{2\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$ → direction Cosines

$\cos\gamma = \dfrac{\sqrt{3}}{2\sqrt{2}}$

$\alpha = \cos^{-1} \dfrac{1}{2\sqrt{2}}$

$\beta = \cos^{-1} \dfrac{-1}{\sqrt{2}} = \dfrac{3\pi}{4}$



$\gamma = \cos^{-1} \dfrac{\sqrt{3}}{2\sqrt{2}}$

Example: If α, β, γ are direction angles of $\vec{a}$ s.t. $\alpha = \frac{\pi}{4}$, $\beta = \frac{2\pi}{3}$, Find all possible values of γ

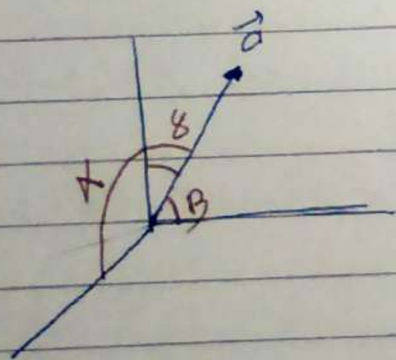
Soln: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

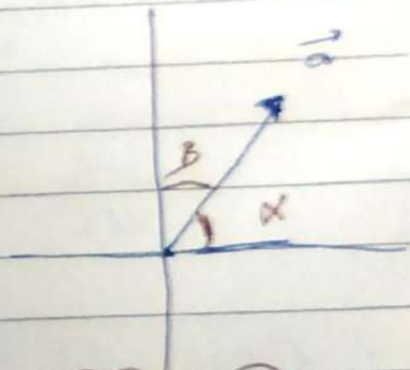
$\cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{2\pi}{3}\right) + \cos^2\gamma = 1$

$\frac{1}{2} + \frac{1}{4} + \cos^2(\gamma) = 1 \implies \cos^2\gamma = \frac{1}{4}$

$\cos\gamma = \frac{1}{2}$ or $= \frac{-1}{2}$

$\gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$

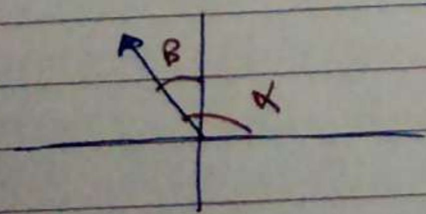




$\alpha + \beta = \frac{\pi}{2}$

$\cos^2\alpha + \cos^2\beta = 1$

#

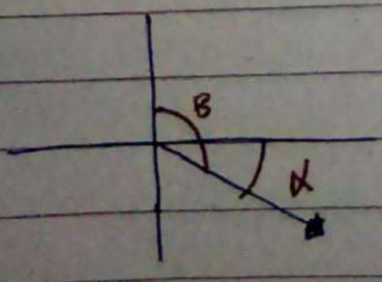


$\cos^2\alpha + \cos^2\left(\frac{\pi}{2} - \alpha\right) = 1$

$\alpha - \beta = \frac{\pi}{2}$

$\cos^2\alpha + \sin^2\alpha = 1$

#

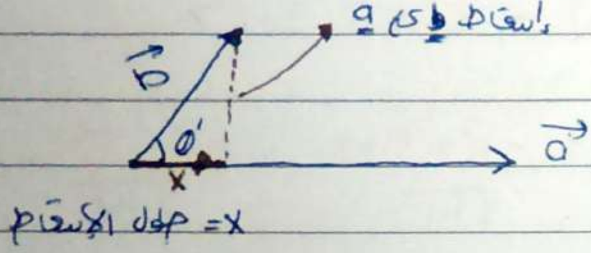


$\cos\left(\frac{\pi}{2} - \alpha\right)$

$= \sin\alpha$

$\beta - \alpha = \frac{\pi}{2}$

#



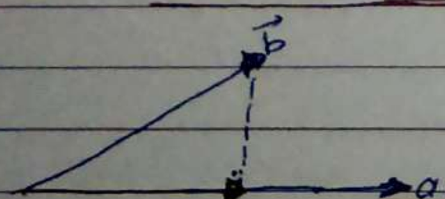
$\cos\theta = \dfrac{X}{|\vec{b}|}$, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$X = |\vec{b}|\cos\theta = \cancel{|\vec{b}|}\,\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,\cancel{|\vec{b}|}}$ $\qquad X = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

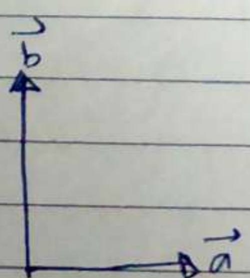
Def: $\boxed{1}$ The scalar projection (proj.) of $\vec{b}$ onto $\vec{a}$ is $\operatorname{Comp}_{\vec{a}} \vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$\boxed{2}$ The vector proj. of $\vec{b}$ onto $\vec{a}$ is:

$\operatorname{proj}_{\vec{a}} \vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

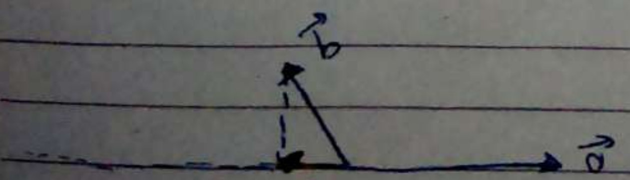


$\operatorname{Proj}_{\vec{a}} \vec{b}$



$\operatorname{proj}_{\vec{a}} \vec{b} = \vec{0}$

$\vec{a}\cdot\vec{b} = 0 \quad (\cos 90^\circ)$



$\operatorname{Proj}_{\vec{a}} \vec{b}$

Example: find the Scalar proj. and vector proj. of $\vec{u} = \langle 1, 1, 2 \rangle$ and $\vec{v} = \langle -2, 3, 1 \rangle$

Sol: $\text{Comp}_{\vec{v}}\, \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \dfrac{-2+3+2}{\sqrt{14}} = \dfrac{3}{\sqrt{14}}$

$\text{Proj}_{\vec{v}}\, \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\, \vec{v} = \dfrac{3}{14} \langle -2, 3, 1 \rangle$

Remark:

$$\underbrace{\left| \text{proj}_{\vec{a}}\, \vec{b} \right|}_{\text{db}} = \underbrace{\left| \text{Comp}_{\vec{a}}\, \vec{b} \right|}_{\text{Abs}}$$

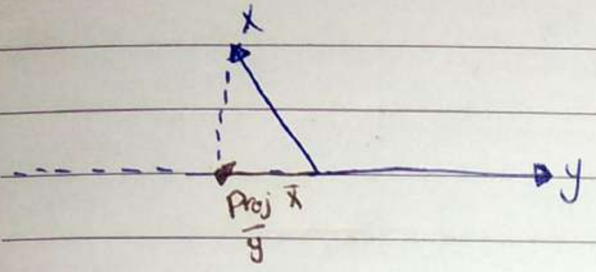
Example: If $\vec{x} \cdot \vec{y} = -3$, $|\vec{y}| = 6$

Find $\left| \text{proj}_{\vec{y}}\, \vec{x} \right|$?! and draw $\vec{x}, \vec{y}$ + $\text{Proj}_{\vec{y}}\, \vec{x}$

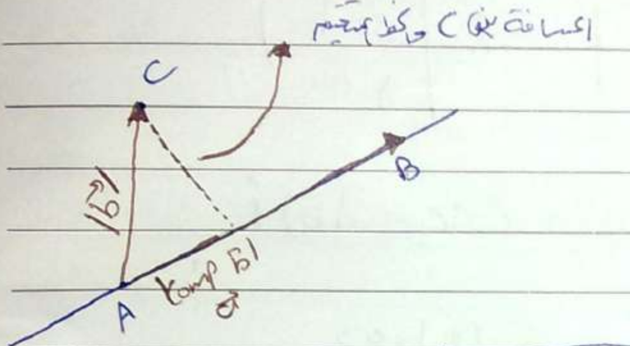
Sol: $\left| \text{proj}_{\vec{y}}\, \vec{x} \right| = \left| \text{Comp}_{\vec{y}}\, \vec{x} \right|$

$$= \left| \frac{\vec{x} \cdot \vec{y}}{|\vec{y}|} \right|$$

$$= \left| \frac{-3}{6} \right| = \boxed{\frac{1}{2}}$$



هاي الحالة لأنه dot product "سالب"



$\vec{a} : \overrightarrow{AB}$

$\vec{b} : \overrightarrow{AC}$

Rule: $(C, \text{line}) = \sqrt{|\vec{b}|^2 - |\text{comp}_{\vec{a}} \vec{b}|^2}$

Example: Find the distance from the pt $(P)\ (-1, 1, 2)$ and the line that pass through the pt. $Q(1, 2, 1)$, $R(0, 0, 1)$

Sol: $\vec{a} = QR = \langle -1, -2, 0 \rangle$ — نثبت أي نقطة من الخط

$\vec{b} = QP = \langle -2, -1, 1 \rangle$

$$\text{dist}\,(P, \text{line}) = \sqrt{|\vec{b}|^2 - |\text{comp}_{\vec{a}} \vec{b}|^2} = \sqrt{6 - \left(\frac{4}{\sqrt{5}}\right)^2}$$

Sec "12.4" The Cross product

let $\vec{u} = \langle a, b, c \rangle$ in V_3 { لأنها تكون في ٣ أبعاد }

$\vec{v} = \langle d, e, f \rangle$

The Cross product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$= +\hat{i} \begin{vmatrix} b & c \\ e & f \end{vmatrix} - \hat{j} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \hat{k} \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= (bf - ec)\hat{i} - (af - dc)\hat{j} + (ae - db)\hat{k}$$

Ex: let $\vec{a} = \langle 3, 2, 1 \rangle$, $\vec{b} = \langle -1, 1, 0 \rangle$
Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$

Sol:

$$\begin{array}{cccc} & + & - & + \\ \vec{a} \times \vec{b} = & \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} \end{array}$$

$$= [(2)(0) - (1)(1)]\, i - [3(0) - (-1)(1)]\, j + [(3)(1) - (-1)(2)]\, k$$

$$= -\hat{i} - \hat{j} + 5\hat{k}$$

$$\begin{array}{cccc} & + & - & + \\ \vec{b} \times \vec{a} = & \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \end{array}$$

$$[(1)(1) - (2)(0)]\hat{i} - [(-1)(1) - (3)(0)]\hat{j} + [(-1)(2) - (3)(1)]\hat{k}$$

$$= \hat{i} + \hat{j} - 5\hat{k}$$

observe that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Properities :

① $\vec{a} \times \vec{b} = -\vec{a} \times \vec{b}$

② $\vec{a} \times \vec{a} = \vec{0}$

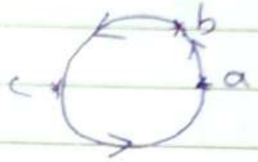
③ $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$

④ $\vec{0} \times \vec{a} = \vec{a} \times \vec{0} = \vec{0} \Rightarrow$ الصفري

⑤ $\vec{a} \times (\vec{b} \pm \vec{c}) = \vec{a} \times \vec{b} \pm \vec{a} \times \vec{c}$

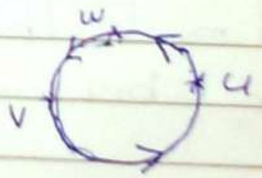
⑥ $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

"لازم نحافظ ع الترتيب!"



Ex: If $\vec{u} \cdot (\vec{w} \times \vec{v}) = -7$, find $\vec{w} \cdot (\vec{u} \times 3\vec{v})$

Sol: $\vec{w} \cdot (\vec{u} \times 3\vec{v}) = 3\vec{w} \cdot (\vec{u} \times \vec{v}) = 3(7) = 21$



Ex: Prove that $(\vec{u} - \vec{v}) \times (\vec{u} + \vec{v}) = 2(\vec{u} \times \vec{w})$

pf: $(\vec{u} - \vec{v}) \times (\vec{u} + \vec{v}) = \vec{u} \times \vec{u} + \vec{u} \times \vec{v} - \vec{v} \times \vec{u} - \vec{v} \times \vec{v}$

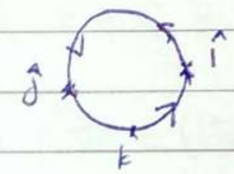
$= \vec{u} \times \vec{v} - (-u \times \vec{v}) = 2(\vec{u} \times \vec{v})$

Remark :

I) $\vec{a}\times(\vec{b}\times\vec{c})$, $(\vec{a}\times\vec{b})\times\vec{c}$ need not be equal

II) $\hat{i}\times\hat{j} = \hat{k}$ $\quad$ $\hat{j}\times\hat{k} = \hat{i}$ $\quad$ $\hat{k}\times\hat{i} = \hat{j}$

$\hat{j}\times\hat{i} = -\hat{k}$ $\quad$ $\hat{k}\times\hat{j} = -\hat{i}$ $\quad$ $\hat{i}\times\hat{k} = -\hat{j}$



To show I) :

$\hat{i}\times(\hat{j}\times\hat{j}) = \hat{i}\times\vec{0} = \vec{0}$

$(\hat{i}\times\hat{j})\hat{j} = \hat{k}\wedge\hat{j} = -\hat{i}$

$\therefore \hat{i}\times(\hat{j}\times\hat{j}) \neq (\hat{i}\times\hat{j})\hat{j}$

Rule :

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

θ, angle between $\vec{a}, \vec{b}$

Remark : $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|}$

Ex : prove that

I) $|\vec{a}\times\vec{b}| = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$ $\Longrightarrow$ متطابقة

Pascal

$\boxed{2}$ $= |\vec{a} \cdot \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \times \vec{b}|^2}$

$\boxed{3}$ $\vec{a} \cdot \vec{b} \underset{\text{موجب}}{=} \sqrt{|\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \times \vec{b}|^2}$, angle between $\vec{a}, \vec{b}$ acute حادة

$\boxed{4}$ $\vec{a} \cdot \vec{b} = -\sqrt{|\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \times \vec{b}|^2}$, angle between $\vec{a}, \vec{b}$ obtuse منفرجة

Prss $\sin^2\theta + \cos^2\theta = 1$

$$\frac{|\vec{a} \times \vec{b}|^2}{|\vec{a}|^2 |\vec{b}|^2} + \frac{(\vec{a} \cdot \vec{b})}{|\vec{a}|^2 |\vec{b}|^2} = 1$$

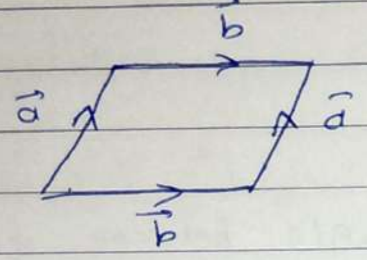
$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

$\boxed{1}$ $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$$

$\boxed{2}$ $(\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \times \vec{b}|^2$

$$(\vec{a} \cdot \vec{b}) = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \times \vec{b}|^2}$$



* يحدد متوازي الاضلاع بمتجهين

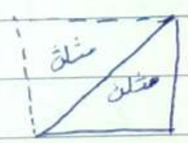
* حساب طول ال Cross product ← بتطلع $\vec{a} \times \vec{b}$ ← يعني الطول
← ع قانون pascal

Rule : [1] The area of the :
* parallelogram determined By the Vectors $\vec{a}, \vec{b}$ is

$$Area = |\vec{a} \times \vec{b}|$$

* triangle determined by $\vec{a}, \vec{b}$ is

$$Area = \frac{1}{2} |\vec{a} \times \vec{b}|$$



مثلثين متطابقين ⇐
مساحتهم متساوية ⇐
يكونو متوازي الاضلاع.

[2] The volume of the ~~Pa~~ Parallepiped
determined by $\vec{a}, \vec{b}, \vec{c}$ is

منشور رباعي

$$Volume = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Remark : $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$
$\vec{c} = \langle c_1, c_2, c_3 \rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} \overset{\oplus}{a_1} & \overset{\ominus}{a_2} & \overset{\oplus}{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\left[(b_2)(c_3) - (c_2)(b_3)\right] - a_2\left[(b_1)(c_3) - (c_1)(b_3)\right]$$

$$+ a_3\left[(b_1)(c_2) - (c_1)(b_2)\right]$$

$\vec{a}\cdot(\vec{b}\times\vec{c})$ called scalar triple of $\vec{a}, \vec{b}, \vec{c}$

Example: find the area of the parallelogram and tringle determind by $\vec{u} = \langle 1, -1, 1 \rangle$
$\vec{v} = \langle 2, 1, 0 \rangle$

Sol: $|\vec{u}\times\vec{v}| = \sqrt{|\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2}$

$$= \sqrt{3(5) - (1)^2} = \sqrt{14}$$

Area of parallelogram = $\sqrt{14}$
Area of tringle = $\frac{\sqrt{14}}{2}$

Example : find the Area of the triangle

with vertices $A(1,-1)$, $B(2,5)$, $C(1,0)$

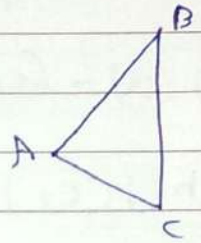
شغالة ب $\mathbb{R}^2$ لو النقطة محددة بإحداثيات رؤوسه

Sol :

لازم نضيف البعد الثالث.

$\vec{AB} = \langle 1, 6, 0 \rangle \Rightarrow$

$\vec{AC} = \langle 0, 1, 0 \rangle$



طبعاً نضيف ثالث لأن كروس برودكت ما بشتغل الا بثلاثة أبعاد

$$Area = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

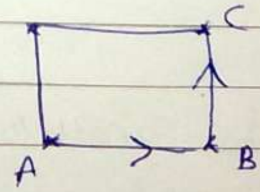
$$= \frac{1}{2} \sqrt{|AB|^2 |AC|^2 - (\vec{AB} \cdot \vec{AC})^2}$$

$$= \frac{1}{2} \sqrt{37(1) - (6)^2} = \frac{1}{2}$$

Ex: Find the Area of parallelogram ABCD with vertices $A(1,-1)$, $B(2,5)$, $C(1,0)$

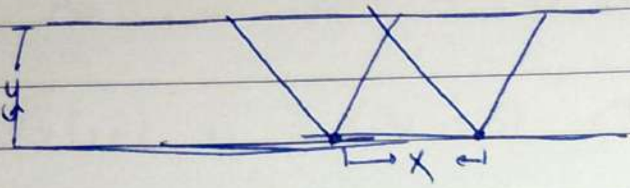
$\vec{AB} = \langle 1, 6, 0 \rangle$

$\vec{BC} = \langle -1, -5, 0 \rangle$



$$Area = \left| \vec{AB} \times \vec{BC} \right|$$

$$= \sqrt{|\vec{AB}|^2 |\vec{BC}|^2 - (\vec{AB} \cdot \vec{BC})^2} = \sqrt{37(26)^2 - (-31)^2}$$



=> المساحة لمتوازيين الاضلاع
نفسه لأنه نقطة البداية والنهاية
نفسها يعني "x" نفسه والارتفاع
نفسه y

Ex: find the Volume of the Parallelepiped determined By:

[1] the Vector $\vec{a} = \langle 6, 3, -1 \rangle$, $\vec{b} = \langle 0, 1, 2 \rangle$

and $\vec{c} = \langle 4, -2, 5 \rangle$

متجاورة

[2] with adjacent edges PQ, PR, PS, where $P(-2, 1, 0)$

حواف

$Q(4, 4, -1)$, $R(-2, 2, 2)$, $S(2, -1, 5)$

Sol: [1] $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix}$

$= 6(5 - -4) - 3(0 - 8) + -1(0 - 4)$
$= 54 + 24 + 4$
$= 82$

Volume: $|\vec{a} \cdot (\vec{b} \times \vec{c})| = |82| = 82$

② $\vec{a} = \overrightarrow{PQ} = \langle 6, 3, -1 \rangle$

$\vec{b} = \overrightarrow{PR} = \langle 0, 1, 2 \rangle$

$\vec{c} = \overrightarrow{PS} = \langle 4, -2, 5 \rangle$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = 82$ by equation 1

$\therefore$ Volume $= 82$

Deff three

1 ~~the~~ points are Colinear if they on the same line

2 four " " Coplanar " " " " plane.

Rules

1 Three points A, B, C are Collinear $\Longleftrightarrow |\overrightarrow{AB} \times \overrightarrow{AC}| = 0$

(2) four " A, B, C, D are Coplanar $\Longleftrightarrow \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0$

Ex: are the points $A \langle 1, 1, 0 \rangle$, $B \langle 2, -1, 1 \rangle$, $C \langle 1, 0, 0 \rangle$ Collinear ?!

نتعد نشتق لي مجموعة صفر

Sol:- $\overrightarrow{AB} = \langle 1, -2, 1 \rangle$

$\overrightarrow{AC} = \langle 0, -1, 0 \rangle$

$\overrightarrow{AB} \times \overrightarrow{AC} = \sqrt{|AB|^2 |AC|^2 - (\overrightarrow{AB} \cdot \overrightarrow{AC})^2}$

$= \sqrt{6 \times 1 - (2)^2} = \sqrt{2} \neq 0 \quad \therefore$ Not Collinear

3,50 ← 2,40 ← ٢ه٢ . ع . م

3,50 ← 2,40 ← ٨/٢٥

Ex8) Are the pts: $P(-2,1,0)$, $Q(4,4,-1)$,

$R(-2,2,2)$, $S(2,-1,5)$ Coplaner ?!

$\overrightarrow{PQ} = \langle 6, 3, -1 \rangle$
$\overrightarrow{PR} = \langle 0, 1, 2 \rangle$
$\overrightarrow{PS} = \langle 4, -2, 5 \rangle$

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 82 \neq 0$$

$\therefore$ Not coplanar

Exercies

find all values of a if exist s.t the pts.

1) $A(a,1,2)$, $B(3,1,5)$, $C(0,1,0)$ are Collinear

2) $A(a,1,2)$, $B(3,1,5)$, $C(0,1,0)$, $D(1,1,1)$ are Coplanar.

Remark:

1) Two vectors are parallel $\Leftrightarrow \vec{a} = k\vec{b}$, k scalar
or $\vec{b} = k\vec{a}$ $k \to$ scalar

2) Two vectors are parallel $\Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$
$|\vec{a} \times \vec{b}| = 0$

3) $\vec{a}, \vec{b}$ perpendicular $\Leftrightarrow \vec{a} \cdot \vec{b} = 0$

Example : $\langle 3,2\rangle \parallel \langle 6,4\rangle$ Since

$\langle 3,2\rangle = \frac{1}{2}\langle 6,4\rangle$

or Since $\left|\langle 3,2\rangle \times \langle 6,4\rangle\right| = \sqrt{13(52) - (26)^2} = 0$

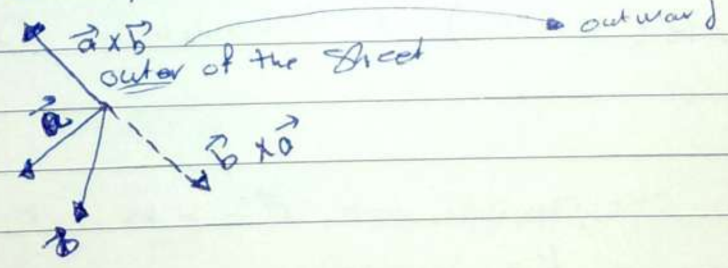
① $\vec{a} = \langle 1, 2, 3\rangle$
$\vec{b} = \langle 3, 6, -9\rangle$

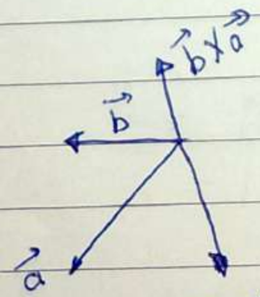
$\vec{a} \nparallel \vec{b}$ Since There is no Scalar k S.T

$\vec{a} = k\vec{b}$

or $|\vec{a} \times \vec{b}| = \sqrt{14(126) - (-12)^2} \neq 0$

Geometric Interpolation of $\vec{a} \times \vec{b}$:





$\vec{a} \times \vec{b}$ into the sheet
inside
inward

Five Apple

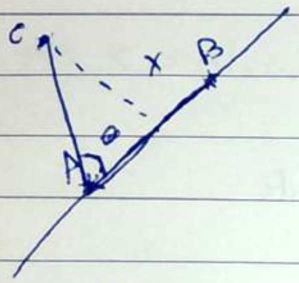
$\vec{a} \times \vec{b}$ perpendecular $\vec{a}$ and $\vec{b}$ also

$\vec{a} \times \vec{b}$ $\perp$ to the plane Containing $\vec{a}, \vec{b}$

وكامودي ع أي متجه داخل هذا المستوى

Section 12.5 equations of lines and planes :

#



لبعد نقطة عن خط مستقيم.

$$\sin\theta = \frac{x}{|\vec{AC}|}$$

$$x = |\vec{AC}| \sin\theta$$

$$= \frac{|\vec{AB}||\vec{AC}| \sin\theta}{|\vec{AB}|}$$

$$= \frac{|\vec{AB} \times \vec{AC}|}{|\vec{AB}|}$$ ⟹ اتجاه العمودي على الخط

Example : Find the distance from the pt. $P(1,2,1)$ and the line through $Q(1,0,1)$, $R(2,1,5)$

Sol $\vec{QP} = \langle 0, 2, 0 \rangle$

$\vec{QR} = \langle 1, 1, 4 \rangle$

$$\text{distance} = \frac{|\overrightarrow{QP} \times \overrightarrow{QR}|}{|\overrightarrow{QR}|}$$

 # Def: The parametric (param.) eqs. of the line "L" that pass through the pt. $A(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$ are

$$x = x_0 + at \qquad , \text{ where } t \in \mathbb{R}$$
$$y = y_0 + bt$$
$$z = y_0 + ct$$

pts on L are
(x_0, y_0, z_0) when $t = 0$
$(x_0 + a, y_0 + b, z_0 + c)$ when $t = 1$
$(x_0 - a, y_0 - b, z_0 - c)$ when $t = -1$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad [\text{symmetric equations of } L]$$

where $a \neq 0$, $b \neq 0$, $c \neq 0$

when $a = 0$

$x = x_0$, $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$, $b \neq 0$, $c \neq 0$

when $b = 0$

$y = y_0$, $\dfrac{x - x_0}{a} = \dfrac{z - z_0}{c}$, $a \neq 0$, $c \neq 0$

When $c = 0$

$z = z_0$, $\frac{x - x_0}{a} = \frac{y - y_0}{b}$, $a \neq 0$, $b \neq 0$

لإيجاد معادلة الخط المستقيم نحتاج

[1] نقطة يمر بها

[2] متجه يوازيه

#Example : find param eqs and symm. eqs of the line

① through the pts. $A(1, 0, 1)$ and parallel to $\vec{u} = \langle 2, -3, 4 \rangle$

② 〃 〃 〃 $A(2, -1, 1)$, $B(3, -1, 2)$

#Sol : [1] param eqs

$x = 1 + 2t$
$y = 0 + -3t$
$z = 1 + 4t$

symm. eqs $\frac{x-1}{2} = \frac{y}{-3} = \frac{z-1}{4}$

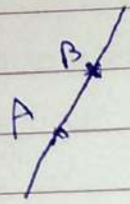
[2] $\vec{v} = \vec{AB} = \langle 1, 0, 1 \rangle$ // line

param eqs
$x = 2 + 1t$
$y = -1 + 0t$
$z = 1 + 1t$

symm. eq
$y = -1$, $x - 2 = z - 1$

Ex: Find the Param and symm eqs of the line that Pass through the pts $A(1,2,3)$, $B(-2,5,7)$. At what pts This line intersects the xy-plane.

Sol:



$\vec{u} = \vec{AB} = \langle -3, 3, 4 \rangle \parallel$ line

pt. on line: A

Param eq. are:

$x = 1 - 3t$

$y = 2 + 3t$

$z = 3 + 4t$

Symm eq are: $\dfrac{x-1}{-3} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$

The line intersects the x-y plane when $z = 0$

$\Rightarrow 3 + 4t = 0 \Rightarrow t = \dfrac{-3}{4}$

$x\Big|_{t=\frac{-3}{4}} = 1 - 3\left(\dfrac{-3}{4}\right) = 1 + \dfrac{9}{4} = \dfrac{13}{4}$

$y\Big|_{t=\frac{-3}{4}} = 2 + 3\left(\dfrac{-3}{4}\right) = 2 - \dfrac{9}{4} = \dfrac{-1}{4}$

$z\Big|_{t=\frac{-3}{4}} = 0$

pt. $\left(\dfrac{13}{4}, \dfrac{-1}{4}, 0\right)$

Five Apple

Remark: [1] let $\vec{u} \parallel L_1$, $\vec{v} \parallel L_2$, $L_1, L_2 \Rightarrow$ lines T

$L_1 \parallel L_2 \Leftrightarrow \vec{u} \parallel \vec{v}$

[2] L_1, L_2 two lines, Then

متقاطعان متخالفان

$L_1 \parallel L_2$ or L_1, L_2 intersected or L_1, L_2 are Skewed

(لا متقاطعان ولا متوازيان)

Ex: Determine whether the two lines L_1, L_2 are parallel, intersected or skewed, If the parallel are they the Same If they intersected find the pts. of intersection.

[1] L_1: $x = 1 - 3t$ $y = 2 + 3t$ $z = 3 + 4t$

L_2: $x = -2 + 3t$ $y = 5 - 3t$ $z = 7 - 4t$

(3) L_1 $x = 2 - 3t$ $y = 2t$ $z = 7$

L_2 $x = 5 + 4t$ $y = 3 - 6t$ $z = 1$

(2) L_1 $x = t$ $y = 3 - t$ $z = 2 + 3t$

L_2 $x = 1 + 2t$ $y = 2 + t$ $z = 5$

(4) L_1 $x = 1 + t$ $y = -2 + 3t$ $z = 4 - t$

L_2 $x = 2t$ $y = 3 + t$ $z = -3 + 4t$

نعم:

Sol :

المتجه الي بوازي اللاين ← من معاملات "t"

$\vec{u} = \langle -3, 3, 4 \rangle \parallel L_1$

$\vec{v} = \langle 3, -3, -4 \rangle \parallel L_2$

$\rightarrow \vec{u} = -\vec{v}$

$\vec{u} \parallel \vec{v} = L_1 \parallel L_2$

To check whether $L_1 = L_2$?!

Take a pt. on L_1 : $A(1, 2, 3)$ (when $t=0$)

What is $t=??$ on L_2 s.t A on L_2 ناخذ يستخدم المعادلة بعملها

$$\left.\begin{aligned} -2+3t &= 1 \\ 5-3t &= 2 \\ 3+4t &= 3 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} t&=1 \\ t&=1 \\ t&=1 \end{aligned}\right\} \rightarrow \text{Yes}$$

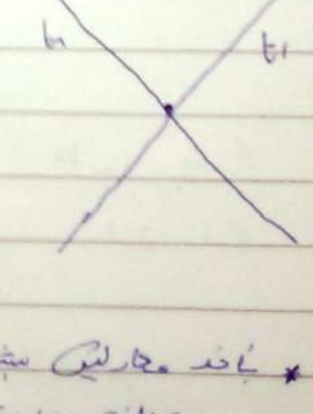
A on L_2 when $t=1$

(2)

$\vec{u} = \langle 1, -1, 3 \rangle \parallel L_1$

$\vec{v} = \langle 2, 1, 0 \rangle \parallel L_2$

$\vec{u} \nparallel \vec{v} \Rightarrow L_1 \nparallel L_2$ (not parallel)



suppos L_1, L_2 intersected

$t_1 = 1 + 2t_2$

$3 - t_1 = 2 + t_2$

$2 + 3t_1 = 5$

* بأخذ معادلتين بستخدم كلاهم وطلع t_1 , t_2 ← نعوض نعالة في المعادلة الثالثة اذا تحققت يعني في نقاط تقاطع فاصعي ما في تقاطع.

$t_1 - 2t_2 = 1 \rightarrow$ ①
$t_1 - t_2 = -1 \rightarrow$ ②
$3t_1 = 3 \rightarrow$ ③

take ② ③

$3t_1 = 3$ $\qquad \boxed{t_1 = 1}$

$t_1 - t_2 = -1$

$-1 - t_2 = -1$ $\qquad \boxed{t_2 = 0}$

Check ① :- $t_1 - 2t_2 \overset{?}{=} 1$

$1 - 2(0) \overset{?}{=} 1$ (yes) $\Leftrightarrow$ L_1, L_2 intersected when

$t_1 = 1$ on L_1

$t_2 = 0$ on L_2

To find the pt. of intersected substitute

$t_1 = 1$ in L_1 (or $t_2 = 0$ in L_2) pt. $(1, 2, 5)$

(3) $\vec{u} = \langle -3, 2, 0 \rangle \parallel L_1$

$\vec{v} = \langle 9, -6, 0 \rangle \parallel L_2$

$\vec{v} = -3\vec{u} \Rightarrow \vec{u} \parallel \vec{v} \Rightarrow L_1 \parallel L_2$

Take A $(2, 0, 7)$ when $t = 0$ on L_1

$x \Rightarrow 5 + 9t = 2 \Rightarrow t = \frac{-1}{3}$

$y \Rightarrow 3 - 6\left(\frac{-1}{3}\right) = 5 \neq 0 \Rightarrow t = \frac{-1}{2}$ مش متساوية t ,t

$\therefore$ A not on L_2

$\therefore L_1 \neq L_2$ (there are not the same)

Five Apple

Scanned by CamScanner

(4) $\vec{u} = \langle 1, 3, -1 \rangle \quad // \, L_1$

$\vec{v} = \langle 2, 1, 4 \rangle \quad // \, L_2$

$\vec{u} \neq \vec{v} \Rightarrow L_1 \not\parallel L_2$

Suppose L_1, L_2 intersected

$$\left.\begin{aligned} 1 + t_1 &= 2t_2 \\ -2 + 3t_1 &= 3 + t_2 \\ 4 - t_1 &= -3 + 4t_2 \end{aligned}\right\} \quad \begin{aligned} t_1 - 2t_2 &= -1 \to ① \\ 3t_1 - t_2 &= 5 \to ② \\ -t_1 - 4t_2 &= -7 \to ③ \end{aligned}$$

Take ①, ③

$$\begin{aligned} t_1 - 2t_2 &= -1 \\ -t_1 - 4t_2 &= -7 \\ \hline -6t_2 &= -8 \end{aligned}$$

$$\boxed{t_2 = \frac{4}{3}}$$

$$t_1 - 2\left(\frac{4}{3}\right) = -1$$

$$t_1 - \frac{8}{3} = -1$$

$$t_1 = -1 + \frac{8}{3} = \frac{5}{3}$$

② $3\left(\frac{5}{3}\right) - \frac{4}{3} \overset{??}{=} 5$

$$\frac{15}{3} - \frac{4}{3} \overset{??}{=} 5$$

$\frac{11}{3} \neq 5 \quad \therefore L_1, L_2$ not intersected

L_1, L_2 skewed

Remark Let L_1, L_2 be intersected lines

$\vec{u} \parallel L_1$, $\vec{v} \parallel L_2$ $\Rightarrow$ the angle θ between L_1, L_2 is

the angle between $\vec{u}, \vec{v}$ $\rightarrow$ $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|u|\,|v|}$

Def: The eq of the plane that pass through the pt. $A(x_0, y_0, z_0)$ and has a normal vector

$\vec{n} = \langle a, b, c \rangle$

is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

* نقطة على المستوى وناظم على المستوى ⇐ نشكل آخر معادلة.

Ex: find the eq. of the plane through the pt. $A \langle 1, 0, 3 \rangle$ and with normal $\vec{n} = \langle 4, -2, 0 \rangle$

$$4(x - 1) - 2(y - 0) + 0(z - 3) = 0$$

$$4x - 2y = 4$$

Normal ⇐ من معاملات x, y, z

Example: Find the eq. of the plane that pass through the pts. A(1,3,2), Q(3,-1,6), R(5,2,0). Find the plane intercepts and sketch this plane:

Sol:

$AQ = \langle 2,-4,4 \rangle$

$AR = \langle 4,-1,-2 \rangle$

$$\vec{n} = AQ \times AR = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = (12)i + 20j + 14\hat{k}$$

The eq. of the plane

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

Intercepts:

x_intercepts: when $y=0$, $z=0$

$12x - 12 - 60 - 28 = 0$

$12x = 100 \implies x = \frac{100}{12}$

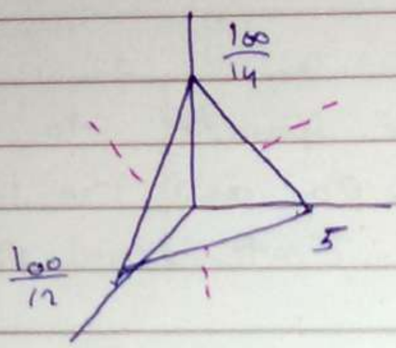
y-intercepts: when $x=0$, $z=0$

$-12 + 20y - 60 - 28 = 0$ $\quad y = \frac{100}{2} = 5$

x-intercepts: when $x=0$, $y=0$

$-12 - 60 + 14z - 28 = 0$

$z = \frac{100}{14}$



Example:

Find the pt at which the line

L_1 $x = 2+3t$, $y = -4t$, $z = 5$

intersects the plane $4x + 5y - 2z = 14$

Sol: $4(2+3t) + 5(-4t) - 2(5) = 14$

$8 + 12t - 20t - 10 = 14$

$-2 - 8t = 14$

$-8t = 16$ $\boxed{t = -2}$

$x = 2 + 3(-2) = -4$

$y = -4(-2) = 8$

$z = 5$

$\Rightarrow$ pt of intersection is $(-4, 8, 5)$

Example: let p_1, p_2 be two plane:

p_1 : $x + z = 1$

p_2 : $y = z$

[1] Find Param eqs of the line of intersection of the planes p_1, p_2

[2] Find the eq of the plane parallel to the line of intersection of p_1, p_2 and pass through the pt. $A(1, 1, 2)$

$\boxed{3}$ Find the eq of the plane parallel to both the line of intersection of P_1, P_2 and the line:
L: $x = 1$, $y = 3 - 2t$, $z = t$, and pass $A(1, 1, 2)$

Sol: intersection of P_1, P_2:
$y = z$, $x + z = -1 \implies x + y = -1$

Take $x = 0 \to y = z$, $z = -1$, $y = -1$
أنا بعوض [illegible]

$B(0, -1, -1)$

Take $x = 1 \implies y = ?$, $z = -2$, $y = -2$
$C(1, -2, -2)$

B, C pts on the line of intersection

$\boxed{1}$ $\vec{v} = \overrightarrow{BC} = \langle 1, -1, -1 \rangle$ // line of intersection
param eqs. of line of intersection

$x = 0 + t$
$y = -1 - t$
$z = -1 - t$

$\boxed{2}$ required plane is P:
$\vec{v}$ in (1) parallel to P

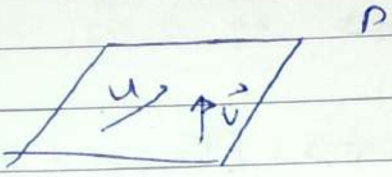
$\overrightarrow{AB} = \langle -1, 2, 3 \rangle$
$\overrightarrow{AC} = \langle 0, -3, -4 \rangle$

$$\oplus \quad - \quad \oplus$$

$$\vec{u} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 3 \\ 0 & -3 & -4 \end{vmatrix}$$

$$= 1i - 4j + 3k$$

$\vec{u} // P$



$n = \vec{u} \times \vec{v}$

$$\begin{vmatrix} i & j & k \\ 1 & -4 & 3 \\ 1 & -1 & -1 \end{vmatrix} = 7i + 4j + 3\hat{k}$$

the eq of plane P is

$$7(x-1) + 4(y-1) + 3(z-2) = 0$$

$\boxed{3}$ vectors $\vec{V} = \langle 1,-1,-1 \rangle$ from 1

$\vec{W} = \langle 0,-2,1 \rangle$ from the line L

$\vec{V}, \vec{w}$ parallel to plane

$$\vec{V} \times \vec{W} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 0 & -2 & 1 \end{vmatrix} = -3\hat{i} - \hat{j} - 2k$$

eq: $-3(x-1)-(y-1)-2(z-2)=0$

Remark: let P_1, P_2 be two planes

$\vec{n_1} \perp P_1$, $\vec{n_2} \perp P_2$

sealed planes

[1] $P_1 // P_2 \Leftrightarrow \vec{n_1} // \vec{n_2}$

[2] $P_1 \nparallel P_2 \Leftrightarrow P_1, P_2$ intersected

The angle θ between P_1, P_2 is

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\ |\vec{n_2}|}$$

Rule: If $A(x_0, y_0, z_0)$ is a pt and

$P: ax+by+cz+d=0$

The distance from the pt. A to P

is dist. $(A, P) = \dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

* قانون المسافة بين نقطة , plane

Example: Find the distance from the pt $P(1,1,2)$ and the plane $2x-4y+z=3$

Sol: dist. $= \dfrac{|2(1)-4(1)+2-3|}{\sqrt{4+16+1}} = \dfrac{3}{\sqrt{21}}$

Remark : P_1 , P_2 two plane

$\boxed{1}$ $P_1 \not\parallel P_2 \Rightarrow$ dist. $(P_1, P_2) = 0$

$\boxed{2}$ $P_1 \parallel P_2 \Rightarrow$ dist. $(P_1, P_2) =$ dist (A, P_2)
where A pt. on P_1

Example :

① P_1 : $x + y = 2$ $\quad$ P_2 : $y - 3z = 0$

$\vec{n_1} = \langle 1, 1, 0 \rangle$ $\quad$ $\vec{n_2}$: $\langle 0, 1, -3 \rangle$

$\vec{n_1} \not\parallel \vec{n_2} \Rightarrow P_1, P_2$ intersected

$\therefore$ dist. $(P_1, P_2) = 0$

② P_1 : $x = 3y$ $\quad$ P_2 : $-2x + 6y = 1$

$\vec{n_1} = \langle 1, -3, 0 \rangle$ $\quad$ $\vec{n_2}$: $\langle -2, 6, 0 \rangle$

$\vec{n_2} = -2\vec{n_1} \Rightarrow \vec{n_1} \parallel \vec{n_2} \Rightarrow P_1 \parallel P_2$

A point on P_1 : $y = 1$, $x = 3$, $z = 0$

$A = (3, 1, 0)$ on P_1

dist. $(P_1, P_2) =$ dist $(A, P_2) = \dfrac{|-2(3) + 6(1) - 1|}{\sqrt{4 + 36}} = \dfrac{1}{\sqrt{40}}$

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Remark: L_1, L_2 two lines

[1] If L_1 intersects $L_2 \Rightarrow$ dist $(L_1, L_2) = 0$

[2) $L_1 // L_2 \Rightarrow$ dist. (L_1, L_2) = dist (A, L_2)

where A pt. on L_1

Example: Find the distance between L_1, L_2 where:

L_1 : $x = 2-3t$ $\quad y = 2t$, $z = 4$

L_2 : $x = 6t$ $\quad y = 1-4t$ $\quad z = 5$

Sol: $\vec{u} = \langle -3, 2, 0 \rangle // L_1$

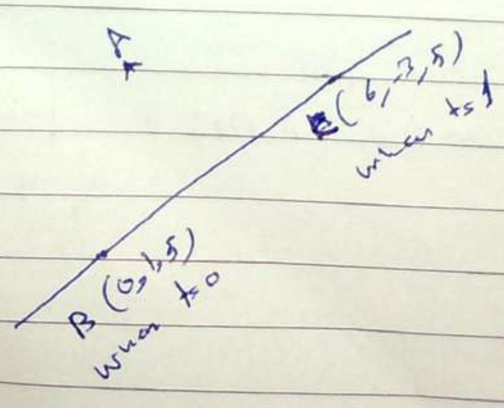
$\vec{v} = \langle 6, -4, 0 \rangle // L_2$

$\vec{u} // \vec{v} \Rightarrow L_1 // L_2$

A pt on L_1 is $A(2, 0, 4) \Rightarrow$ when $t = 0$

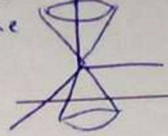
dist (L_1, L_2) = dist (A, L_2)

$$= \frac{|\vec{AB} \times \vec{BC}|}{|\vec{BC}|}$$

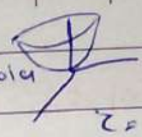


① 

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

② 

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

③ 

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Section 12.6 Cylinders and Quadric surfaces

cylinders: surfaces obtained by moving a curve in the direction of a fixed axis.

Example: $y = x^2$, $x^2 + y^2 = 4$ cylinders

Quardic surface: is the graph of a 2^{th} degree eq in the variables x, y, z

* معادلة أي أس فيها = ٢

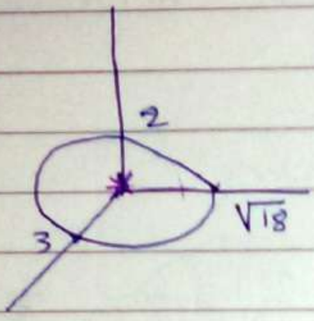
Example: $x^2 - 3y^2 - 5z = 3$, $3xy + 5z^2 - z + 3y = 0$ } quadric Surfaces.

Example: Identify (give the name) and sketch the surface

1) $4x^2 + 2y^2 + 9z^2 = 36$
2) $2x^2 + 4y^2 + 9z^2 = 36$
3) $x^2 - 6x + 3y^2 + z^2 - 10z + 25 = 0$
4) $4x^2 - y^2 + z^2 = 6$
5) $y^2 + x^2 - z^2 = 7$
6) $2x^2 - \frac{y^2}{2} - 16z^2 = 8$
7) $y^2 = x^2 + z^2 + 1$
8) $y^2 - x^2 + z^2 = 0$
9) $z^2 = x^2 + y^2$
10) $z = \sqrt{x^2 + y^2}$
11) $z = -\sqrt{x^2 + y^2}$
12) $y^2 = 2x^2 + 3z^2$
13) $20x - y^2 - z^2 = 0$
14) $20x + y^2 + z^2 = 0$
15) $x^2 + 6x + y + z^2 + 10 = 0$
16) $x^2 + 6x - y + z^2 + 10 = 0$
17) $z = y^2 - x^2$

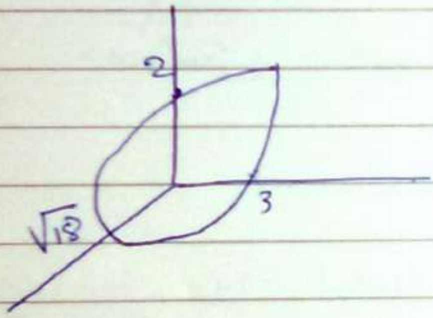
Sol of [1] $\frac{x^2}{9} + \frac{y^2}{12} + \frac{z^2}{4} = 1$ ellipsoide

* مركز يأتي من تصغير التربيعات.
* بنحدد كل آية كمية حسب الرقم الأكبر في المقام.



intercepts ⇒ من التصفير

[2] eq. $\frac{x^2}{18} + \frac{y^2}{9} + \frac{z^2}{4} = 1$ ellipsoide

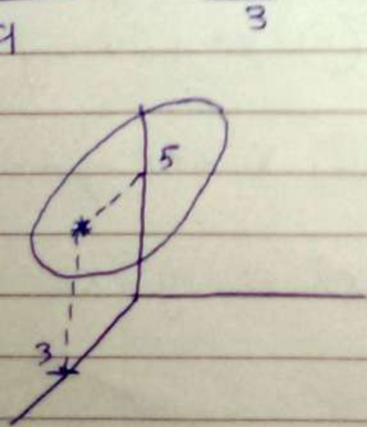


[3] $x^2 - 6x + 9 + 3y^2 + z^2 - 10z + 25 = 9 + 25 - 25$

$(x-3)^2 + 3y^2 + (z-5)^2 = 9$

$\frac{(x-3)^2}{9} + \frac{y^2}{3} + \frac{(z-5)^2}{9} = 1$ ellipsoide

* ٣ تربيعات
* ثابت بيجي لوحده
* كله موجب



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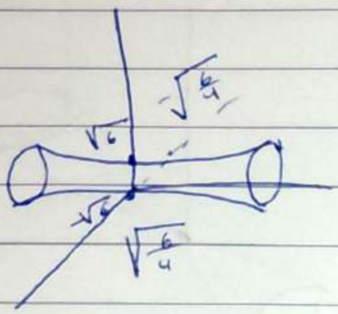
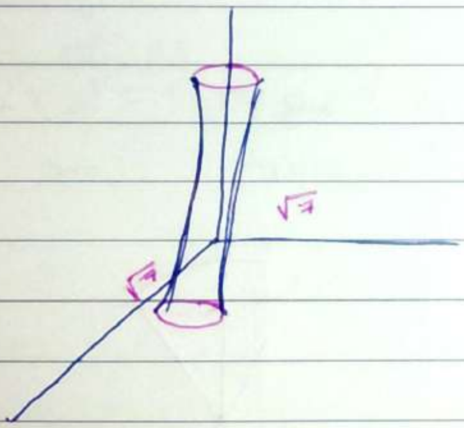
4) $\frac{x^2}{(\frac{6}{4})} - \frac{y^2}{6} + \frac{z^2}{6} = 1$ Hyperbolid of one sheet

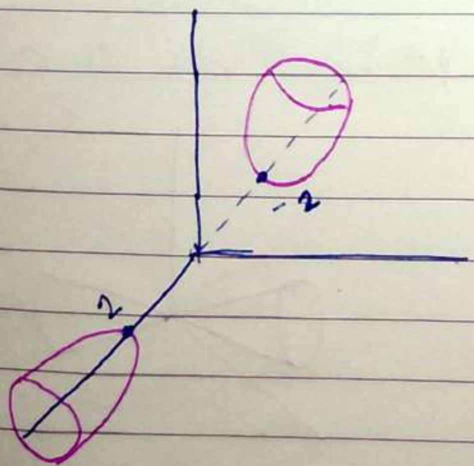


*هاي مقطع مهم.

5) Hyperbolid of one sheet

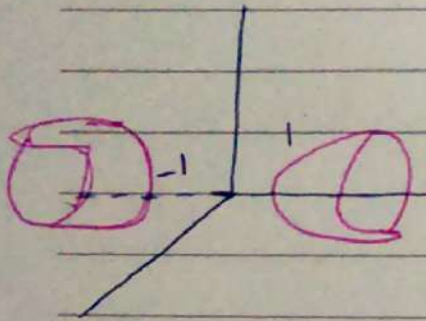


6) $\frac{x^2}{4} - \frac{y^2}{16} - 2z^2 = 1$ Hyperboloid of two sheets.



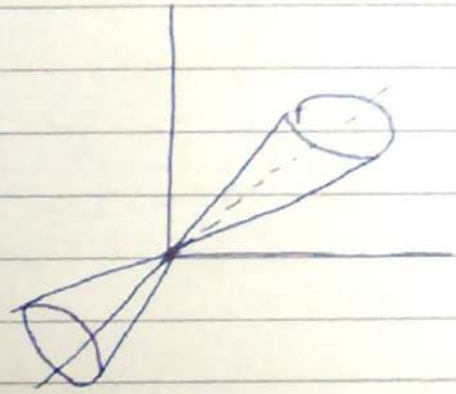
7) $y^2 - z^2 - x^2 = 1$

Hyperbolid of two Sheets



8) $x^2 = y^2 + z^2$

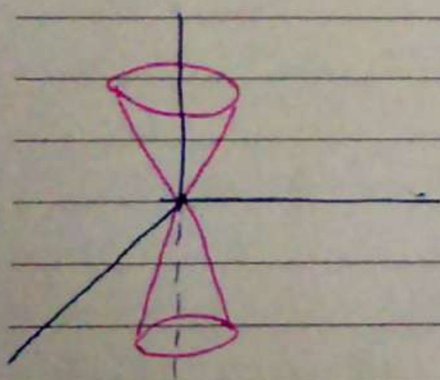
elliptic Corn



ما z راكي لنقاط القاطع ⇐ x بين بيتقاطع

بال origin

9) $z^2 = x^2 + y^2$



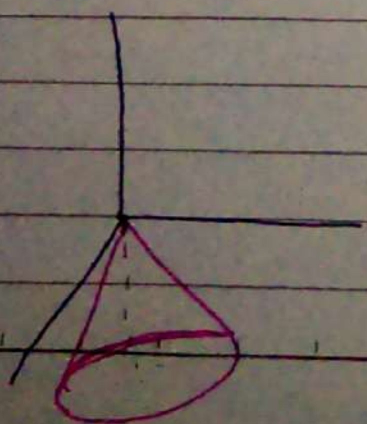
10) eq $\Rightarrow$ $z^2 = x^2 + y^2$

elliptic Cone.

دائماً صفر او اكبر موجب ⇐ z لأنه موجبة

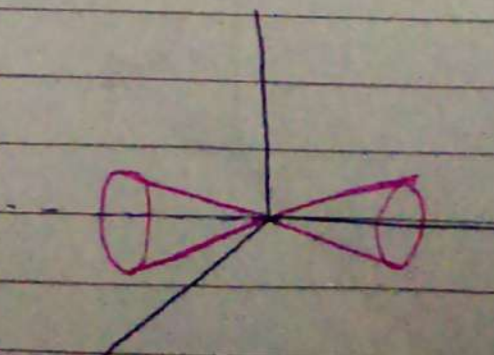
11) $z = -\sqrt{x^2 + y^2}$

elliptic Corn



$z \leq 0$

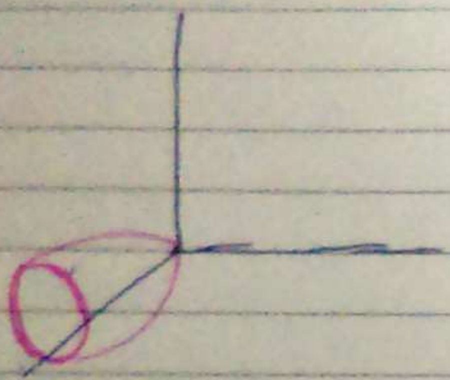
12) elliptic Corn



Five Apple

13 $20x = y^2 + z^2$

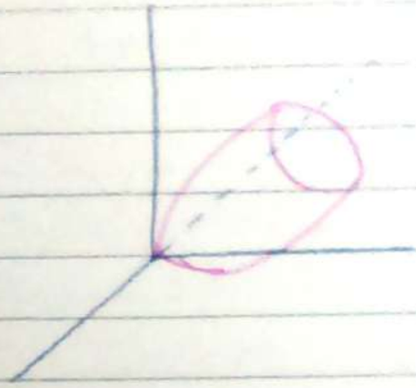
elliptic Parabolid



Since $x \geq 0$

14 $20x = -(y^2 + z^2)$

elliptic parabolod



$x \leq 0$

15 ~~x^2...+9~~

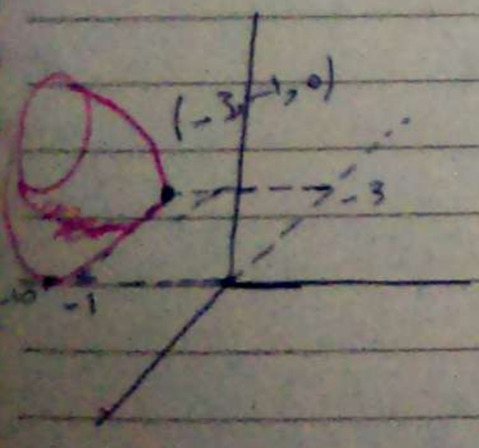
$(x+3)^2 + y + z^2 = -10 + 9$

$y = -[(x+3)^2 + z^2] - 1 \Rightarrow$ مفتوح

$y + 1 = -[(x+3)^2 + z^2]$ لتحت

بجهة محور y

Elliptic parabloid



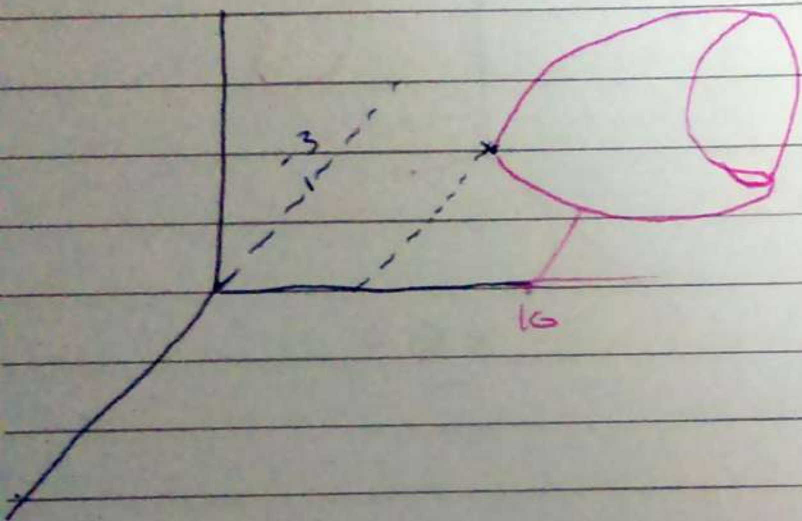
$-(y+1) = 9$

$y = -10 \rightarrow$ y-intercept

$\boxed{16}$ $(x+3)^2 - y + z^2 = -10 + 9$ Elliptic Paraboloid

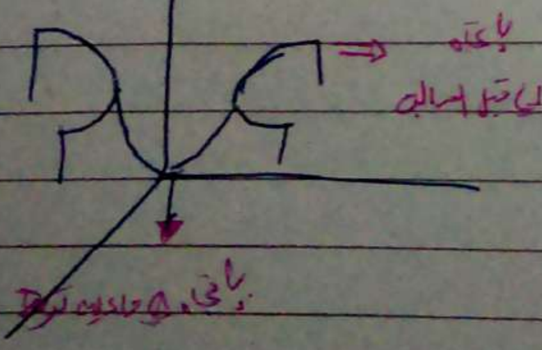
$(x+3)^2 - y + z^2 = -1$

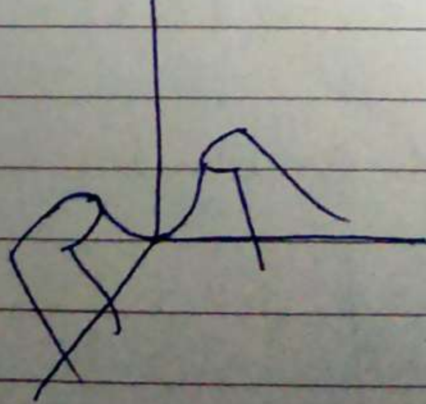
$y - 1 = (x+3)^2 + z^2$



$\boxed{17}$ HYperbolic Parabolid

$z = x^2 - y^2$





Ch. 14: Partial Derivatives

Sec 14.1: Function of several variables

Example: $f(x,y) = \ln\left(y - \sqrt{x^2+y^2}\right)$

$\text{Domain}(f) = \left\{ (x,y) : y - \sqrt{x^2+y^2} > 0 \right\}$

$= \left\{ (x,y) : y > \sqrt{x^2+y^2} \right\}$

$1 > \sqrt{0^2+1^2}$

$(0,1) \notin \text{Dom}(f)$

$(3,4) \notin \text{Dom}(f)$ since $4 \not> \sqrt{3^2+4^2}$

$(0,4) \notin \text{Dom}(f)$ since $4 \not> \sqrt{0^2+4^2} = 4$

Example: $f(x,y,z) = \dfrac{1}{x^2-y^2+z-3}$

$\text{Dom}(f) = \left\{ (x,y,z) : x^2-y^2+z-3 \neq 0 \right\}$

$(1,1,3) \notin \text{Dom}(f)$ since $1^2-1^2+3-3 = 0$

$(1,1,1) \in \text{Dom}(f)$ since $1^2-1^2+1-3 = -2 \neq 0$

$f(1,1,1) = \dfrac{1}{-2} = \dfrac{-1}{2}$

Example : $g(x,y) = \sqrt{x^2-y^2}$

$\text{Dom}(g) = \{(x,y) : x^2-y^2 \geq 0\}$

$(1,2) \notin \text{Dom}$ ~~Sine~~ Since $1^2-2^2 = -3 \not\geq 0$
$(2,1) \in \text{Dom}$ since $2^2-1^2 = 3 \geq 0$

Example : $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2-y^2} & , |x| \neq |y| \\ 3 & , |x| = |y| \end{cases}$

$\text{Dom}(f) = \{(x,y) : x,y \in \mathbb{R}\} = \mathbb{R}^2$

$f(1,-1) = 3$ Since $|1| = |-1|$
$f(3,4) = \dfrac{\sin(3^2+4^2)}{3^2-4^2} = \dfrac{\sin(25)}{-7}$

Example : $g(x,y,z) = \sqrt{1+x^2+y^2+z^2}$

$\text{Dom}(g) = \{(x,y,z) : x,y,z \in \mathbb{R}\} = \mathbb{R}^3$

$f(1,-1,3) = \sqrt{1+1^2+(-1)^2+2^2} = \sqrt{7}$

Sec 14.2 Limits and Continuity

A Curve c in $\mathbb{R}^3$ (or $\mathbb{R}^2$) is given by $x = f(t)$,

$y = g(t)$, $z = h(t)$, $t \in [a,b]$

A pt. on C is $(f(t), g(t), h(t))$, $t \in [a,b]$

Example: C: $x = t+1$, $y = t^2 - 2$

Curve in $\mathbb{R}^2$

z is zero

[1] $(1, -2)$ pt. on C when $t =$ zero

[2] $(0, -1)$ pt. on C when $t = -1$

[3] $(2, 2)$ not a pt. on C because there is no t s.t.

$t + 1 = 2 \Rightarrow t = 1$

$t^2 - 2 = 2 \Rightarrow t = \pm 2$

Example: C: $x = t$, $y = t^2$, $z = t+1$

Curve in $\mathbb{R}^3 \Rightarrow$ $z \neq 0$

Def: let C $x = f(x)$, $y = g(t)$, $z = h(t)$ curve in $\mathbb{R}^3$ pass through a pt. (x_0, y_0, z_0) when $t = t_0$

$$\lim_{(x,y,z) \to (x_0, y_0, z_0)} F(x, y, z) = L \text{ exist } \Leftrightarrow \lim_{(x,y,z) \to (x_0, y_0, z_0)} F(x, y, z) = L$$

$L \in \mathbb{R}$ along any curve C

$$\Leftrightarrow \lim_{t \to t_0} F(f(t), g(t), h(t)) = L$$

Also:

$$\lim_{(x,y,z) \to (x_0, y_0, z_0)} F(x, y, z) = \text{does not exist (DNE)}$$

$\Leftrightarrow \lim F(x, y, z)$ along C_1 $\neq \lim F(x, y, z)$ along C_2

C_1, C_2 curves pass (x_0, y_0, z_0)

Examples: Find the limit if it exists

1) $\lim_{(x,y,z)\to(1,-1,2)} e^{-xyz}\cos(x-y)$

2) $\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y-6y^2}{x^2+3y}$

3) $\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(2x^2+2y+z^2)}{2x^2+2y+z^2}$

4) $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2y^2)}{x-y}$

5) $\lim_{(x,y)\to(0,0)} \frac{(2x-3y+1)^{13}-1}{(2x-3y-2)^{13}+8}$

6) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

Sol: $\lim_{(x,y,z)\to(1,-1,3)} e^{-xyz}\cos(x-y) = e^{-1(-1)(2)}\cos(1--1) = e^2\cos 2$

* تعويض مباشر

2) $\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y-6y^2}{x^2+3y} = \left(\frac{0}{0}\right)$

$\lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+3y)}(x^2-2y)}{\cancel{(x^2+3y)}} = \lim_{(x,y)\to(0,0)} (x^2-2y) = 0$

* تحليل

3) $\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(2x^2+2y+z^2)}{2x^2+2y+z^2} \quad \left(\frac{0}{0}\right)$

$W = 2x^2 + 2y + z^2 \quad \to$ when $(x, y, z) \to (0, 0, 0)$

Then $W \to 0$

$$\Rightarrow \lim_{W \to 0} \frac{\sin W}{W} = \boxed{1}$$

$\boxed{4}$ $$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2 - y^2)}{x - y} = \frac{0}{0}$$

$$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2-y^2) * (x+y)}{(x-y)(x+y)} = \lim_{(x,y)\to(0,0)} \frac{\tan(x^2-y^2)}{(x^2-y^2)} * (x+y)$$

$$= 1 * 0 = \boxed{0}$$

$\boxed{5}$ $$\lim_{(x,y)\to(0,0)} \frac{(2x-3y+1)^{13}-1}{(2x-3y-2)^{13}+8} = \frac{0}{(-2)^{13}+8} = \boxed{0}$$

$\boxed{6}$ $$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2+y^2} = \left(\frac{0}{0}\right) \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = \boxed{0}$$

$\boxed{7}$ $$\lim_{(x,y)\to(1,-1)} \frac{(2x+y)^5 - 1}{(4x+2y)^5 - 32} \quad \left(\frac{0}{0}\right)$$

$$\lim_{(x,y)\to(1,-1)} \frac{(2x+y)^5-1}{(2((2x+y)^5)-32} \quad \boxed{\text{II}} \quad \frac{\cancel{(2x+y)^5-1}}{32\,[(2x+y)^5-1]} = \frac{1}{32} \to$$ صح
تعويض

فرض
$W = 2x+y \iff (x,y) \to (1,-1) \Rightarrow W \to 1$

$$\lim_{W\to 1} \frac{W^5 - 1}{(2W)^5 - 32} = \lim_{W\to 1} \frac{5W^4}{5(2W^4)(2)} = \frac{1}{2^5} = \frac{1}{32}$$

طالما بقدر أجيب نظيرها للحل
يبقى صح ماشي موضوع تعويض ده

Five Apple

5) $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$ → Polar Substitution
$(x,y) \to (0,0)$
$r \to 0^+$

$\lim_{r\to 0} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2}$

عن طريق العامل المشترك

$= \lim_{r\to 0^+} \frac{r^4(\cos^4\theta + \sin^4\theta)}{r^4}$

$= \lim_{r\to 0^+} r^2(\cos^4\theta + \sin^4\theta) = \boxed{0}$

Example : Find the limit if it exist.

1) $\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2(2y)}{2x^2+y^2} = (0/0)$

C_1 : $x = t$, $y = 0$ pass $(0,0)$ when $t = 0$

$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2(2y)}{2x^2+y^2} = \lim_{t\to 0} \frac{t^2+\sin^2(0)}{2t^2+0^2} = \boxed{\frac{1}{2}}$

C_2 : $x = 0$, $y = t$ pass $(0,0)$ when $t = 0$

$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 2y}{2x^2+y^2} = \lim_{t\to 0} \frac{\sin^2(2t)}{t^2} = 4$

$\therefore \lim_{\text{along } C_1} f \neq \lim_{\text{along } C_2} f$ ∴ the limit d.n.e

$C_1 = x = t+1$, $y = 0$ Pass $(1,0)$ when $t = 0$

2) $\lim_{(x,y)\to(1,0)} \frac{x^2-2x+1-y^2}{(x-1)^2+y^2} = \lim_{t\to 0} \frac{(t+1)^2-2(t+1)+1-0}{(t+1-1)^2+0}$

$\overset{L}{=} \lim_{t\to 0} \frac{2(t+1)-2}{2t} = \boxed{1}$

Five Apple

$C_2: \quad x = 1, \quad y = t \quad$ pass $(1,0)$ when $t = 0$

$$\lim_{(x,y)\to(1,0)} \frac{x^2-2x+1-y^2}{(x-1)^2+y^2} = \lim_{t\to 0} \frac{-t^2}{t^2} = -1$$

$\therefore$ the limit d.n.e

[3] $$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2+4y^2+9z^2} = \frac{0}{0}$$

$C_1: \quad x = t \quad y = 0, \quad z = 0 \quad$ pass $(0,0,0)$ when $t = 0$

$$\lim_{\text{along } C_1} f = \lim_{t\to 0} \frac{0}{t^2} = 0$$

$C_2: \quad x = 0, \quad y = t, \quad z = t \quad$ pass $(0,0,0)$ when $t = 0$

$$\lim_{\text{along } C_2} f = \lim_{t\to 0} \frac{t^2}{13t^2} = \frac{1}{13}$$

$\therefore$ the limit d.n.e.

[4] $$\lim_{(x,y)\to(0,0)} \frac{x^{2/3}\, y^2}{x^2+y^2}$$

let $x = r\cos\theta$

$y = r\sin\theta$ d.n.e $-1 \le \cos, \sin \le 1$

$$\lim_{r\to 0} \frac{r^{2/3}\cos^{2/3}\theta\, \cancel{r^2}\sin^2\theta}{\cancel{r^2}(1)} = \boxed{0}$$

[6] $$\lim_{(x,y)\to(0,0)} \frac{x^{2/3}\, y^2}{x^2+y^3}$$

$C_1: \quad x = t, \quad y = 0 \quad$ pass $(0,0)$ when $t = 0$

$$\lim_{\text{along } C_1} f = \lim_{t\to 0} \frac{0}{t^2} = \boxed{0}$$

$C_2: \quad x = t^3, \quad y = t^2 \quad$ pass $(0,0)$ when $t = 0$

$$\lim_{\text{along } C_2} f = \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2}$$

$\therefore$ the limit d.n.e

$\boxed{7}$ $$\lim_{(x,y,z)\to(0,0,1)} \frac{xy^2+y^2z-y^2}{x^2+y^4+(z-1)^2}$$

C_1: $x=t$, $y=0$, $z=1$ pass $(0,0,1)$ when $t=0$

$$\lim_{\text{along } C_1} f = \lim_{t\to 0} \frac{0}{t^2} = 0$$

C_2: $x=t^2$, $y=t$, $z=t^2+1$ pass $(0,0,1)$ when $t=0$

$$\lim_{\text{along } C_2} f = \lim_{t\to 0} \frac{t^4+t^2(t^2+1)-t^2}{3t^4} = \lim_{t\to 0} \frac{2t^4}{3t^4} = \boxed{\frac{2}{3}}$$

$\therefore$ The limit d.n.e

Def:

A function $f(x,y)$ is Con'ts at $(a,b) \in Dom(f)$

if $\lim_{(x,y)\to(a,b)} f = f(a,b)$

Examples:

① $f(x,y) = \frac{x^2+y^2}{x^2+y^2+1}$ is Conts on $\mathbb{R}^2$

2) $g(x, y, z) = \dfrac{x^2 - y^2}{e^{xyz}}$ cont's on $\mathbb{R}^3$

3) $h(x, y) = \dfrac{1}{x-y}$ cont on $\mathbb{R}^2 - \{(x, x) : x \in \mathbb{R}\}$

4) $f(x, y) = \dfrac{\sqrt{y - x^2}}{1}$ cont on $\{(x, y) : y - x^2 \geq 0\}$

Example: find a s.t $f(x,y) = \begin{cases} \dfrac{ax^2-y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ -1 & , (x,y) = (0,0) \end{cases}$

Conts at $(0,0)$.

Sol: $\lim\limits_{(x,y)\to(0,0)} f = f(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{ax^2-y^2}{x^2+y^2} = -1$$

$C: x = t , y = 0$ pass $(0,0)$ when $t = 0$

$$\lim_{\text{along } C} \frac{ax^2-y^2}{x^2+y^2} = -1$$

$$\lim_{t\to 0} \frac{at^2}{t^2} = -1 \Rightarrow \boxed{a=-1}$$

Sec 14.3: Partial Derivatives:

The partial derivative of $z = f(x,y)$ with respect to (w.r.t)

1) x at (a,b) is $f_x(a,b) = \lim\limits_{x\to a} \dfrac{f(x,b)-f(a,b)}{x-a}$

$$f_x(a,b) = \frac{\partial f}{\partial x}\bigg|_{(a,b)} = z_x\big|_{(a,b)} = \frac{\partial z}{\partial x}\bigg|_{(a,b)}$$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

Five Apple

(2) y at (a,b) is $f_y(a,b) = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b}$

$$f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \left. z_y \right|_{(a,b)} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

Example : If $z = \sin\left(\frac{x}{1+y^2}\right)$, then

(1) $\frac{\partial z}{\partial x} = \cos\left(\frac{x}{1+y^2}\right) \cdot \left(\frac{1}{1+y^2}\right)$

(2) $z_y = \cos\left(\frac{x}{1+y^2}\right) \cdot \frac{-x(2y)}{(1+y^2)^2}$

(3) $\left. z_y \right|_{(1,-1)} = \cos\left(\frac{1}{2}\right)\left(\frac{2}{4}\right) = \frac{1}{2}\cos\frac{1}{2}$

Example : let $f(x,y,z) = x^3 + e^{2xz} + \ln(xy)$

find $f_{(x)}(1,1,2) + 3 f_y(1,1,2) - 3 f_z(1,1,2) = 4$

Sol : $f_x = 3x^2 + 2z e^{2xz} + \frac{\cancel{y}}{x\cancel{y}}$

$f_x(1,1,2) = 3 + 4e^4 + 1 = 4 + 4e^4$

$f_y = 0 + 0 + \frac{x}{xy}$

$f_y = (1,1,2) = 1$

$f_z = 0 + 2x\,e^{2xz} + 0$

$f_z = (6, 1, 2) = 2e^4$

$w = 4 + 4e^4 + 3(1) - 3(2e^4)$

Example : Find $f_x(0,0)$, $f_y(0,0)$ if it exist where

$f(x,y) = (x^3 - y^3)^{2/3}$

$f(x) = \frac{2}{3}(x^3 - y^3)^{-\frac{1}{3}}(3x^2)$

$= \dfrac{2x^2}{(x^3 - y^3)^{\frac{1}{3}}} = \left(\frac{0}{0}\right)$ فشلت قاعدة الاشتقاق

فنرجع للعام ال lim

$f_x(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{(x^3)^{\frac{2}{3}} - 0}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{x^2}{x} = \boxed{0}$

$f_y(0,0) = \lim\limits_{y \to 0} \dfrac{f(0,y) - f(0,0)}{y - 0} = \lim\limits_{y \to 0} \dfrac{(-y^3)^{\frac{2}{3}} - 0}{y}$

$= \lim\limits_{y \to 0} \dfrac{y^2}{y} = 0$

Example : Find $f_x(0,0)$, $f_y(0,0)$ where $f(x,y) = \sqrt{x^2 + y^4}$

Sol : $f(x) = \dfrac{2x}{2\sqrt{x^2 + y^2}}$, $f_x(0,0) = \frac{0}{0}$ X

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt{x^2} - 0}{x}$$

$$= \lim_{x \to 0} \frac{|x|}{x}$$

صعود متصل
جهتين لا
كالكلمه 1
لأنها متغير
واحد

$$\left.\begin{array}{l} \lim_{x \to 0^+} \frac{x}{x} = 1 \\ \lim_{x \to 0^-} \frac{-x}{x} = -1 \end{array}\right\} \rightarrow \lim_{x \to 0} \frac{|x|}{x} \text{ DNE}$$

$f_x(0,0)$ DNE

$$f_y(0,0) = \text{-----} = \boxed{0}$$

Exercise : find $f_x(0,0)$, $f_y(0,0)$ when $f(x,y) = (x^3 - y^3)^{1/3}$

Remark : $f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$

$f_{xy} = (f_x)_y$ * أول مرة نشتق بالنسبة لـ (x) لنعيد بالنسبة لـ (y)

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$f_{xxyxzx} = \frac{\partial^6 f}{\partial x \partial z \partial x \partial y \partial x^2}$$

Example: find F_{xxyz} if $f(x, y, z) = \sin(3x + yz)$

$f_x = \cancel{3\cos} \quad 3\cos(3x + yz)$

$f_{xx} = -9\sin(3x + yz)$

$f_{xxy} = -9z\cos(3x + yz)$

$f_{xxyz} = -9z(-y\sin(3x + yz)) + -9\cos(3x + yz)$

$= 9yz\sin(3x + yz) - 9\cos(3x + yz)$

thm: let $f(x,y)$ be defined on a disk "D" that contains a pt. (a,b). If f_{xy}, f_{yx} cont. on D then $f_{xy}(a,b) = f_{yx}(a,b)$


D

$* \; f_{xxyxyy} = f_{xxxyyy} = f_{yyyxxx}$ — الترتيب مش مهم في الكالكولس 3

Example: $\dfrac{\partial^{72} xe^{xy}}{\partial^{70}y \, \partial x^2}$

رتبة مشتقة د y	xe^{xy}
1	$x^2 e^{xy}$
2	$x^3 e^{xy}$
⋮	
70	$x^{71} e^{xy}$

Sol: $\dfrac{\partial^{72} xe^{xy}}{\partial x^2 \, \partial^{70} y} \Rightarrow$

$= \dfrac{\partial^2 x^{71} e^{xy}}{\partial x^2}$

$= \dfrac{\partial}{\partial x}\left[y x^{71} e^{xy} + 71x^{70} e^{xy}\right]$

$= \dfrac{\partial}{\partial x}\left[\left(x^{71} y + 71x^{70}\right) e^{xy}\right]$

$$= (x^{71}y + 71x^{70})\,y e^{xy} + e^{xy}(71x^{70}y + (71)(70)x^{69})$$

$$\left.\frac{\partial^{72} xe^{xy}}{\partial x^{2}\,\partial y^{70}}\right|_{(1,0)} = 0 + 1\,(70(71))$$

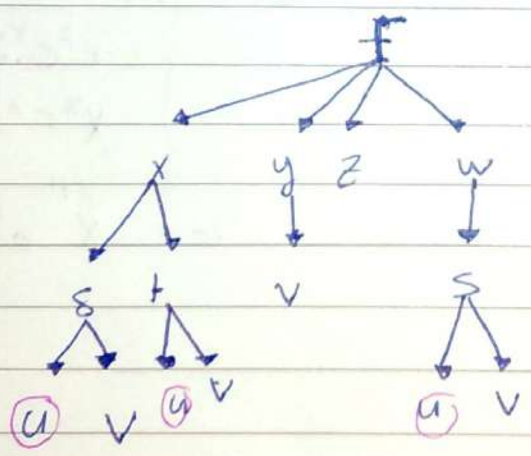
Exercise: find $\dfrac{\partial^{100} f}{\partial y^{40}\,\partial x^{60}}$ where $f(x,y) = x^{50}\sin y + y^{87}\,y^{100}$

Sec 14.5 The Chain Rule

let $f(x, y, z, w)$, $x = x(s,t)$, $y = y(v)$, $w = w(s)$

لم تعين x اقتران بدلالة s,t

$s = s(u,v)$, $t = t(u,w)$



جيد ← الفروع التي تظهر بها u كل الطرق

$$\frac{\partial f}{\partial u} = \cancel{\quad} = f_x \cdot x_s \cdot s_u + f_x \cdot x_t \cdot t_u + f_w \cdot w \cdot s_u$$

لا يجوز كتابة "w" بالسالب

$$\frac{\partial f}{\partial v} = f_x \cdot x_s \cdot s_v + f_x \cdot x_t \cdot t_v + f_y \cdot \frac{\partial y}{\partial v} + f_w \cdot \frac{\partial w}{\partial s} \cdot s_v$$

$$x_u = x_s \cdot s_u + x_t \, t_u$$

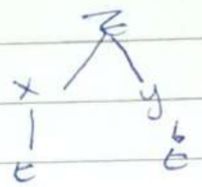
$$f_s = f_x \cdot x_s + f_w \cdot \frac{\partial w}{\partial s}$$

Example : let $z = x^2 y + 3xy^2$

$x = \sin 2t$

$y = \cos 2t$

Find $\left.\frac{\partial z}{\partial t}\right|_{t=0}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy + 3y^2)(2\cos 2t) + (x^2 + 6xy)(-2\sin 2t)$$

$$x\big|_{t=0} = \sin[(2)(0)] = 0$$

$$y\big|_{t=0} = \cos[(2)(0)] = 1$$

$$\frac{dz}{dt} = \left(2(0)(1) + 3(1)^2\right)\left(2\cos 2(0)\right) + \left(0^2 + 6(0)(1)\right)(-2\sin \ldots$$

$$= 6$$

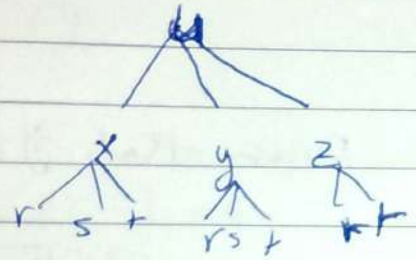
Example: If $u = x^4 + y^2 z^3$

$x = rse^t$

$y = rs^2 e^{-t}$

$z = r^2 \sin t$

Find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$ when $r = 2$, $s = 1$, $t = 0$



$x\Big|_{\substack{r=2\\ s=1\\ t=0}} = 2(1)e^0 = 2$

$y\Big|_{\substack{r=2\\ s=1\\ t=0}} = 2(1)^2(1) = 2$

$z\Big|_{\substack{r=2\\ s=1\\ t=0}} = 0$

$\frac{\partial u}{\partial s} = u_x \cdot x_s + u_y \cdot y_s$

$= 4x^3 re^t + 2yz^3 \cdot 2rse^{-t}$

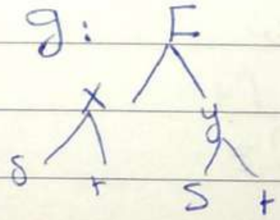
$= 4(2)^3 \cdot 2 \cdot e^0 + 2(2)(0)(2)(1)e^0$

$= 64$

$\frac{\partial u}{\partial t} = u_x \cdot x_t + u_y \cdot y_t + u_z \cdot z_t$

Example: If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ show that

$t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0$



Sol: $f(x,y)$, $x = s^2 - t^2$

$y = t^2 - s^2$

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t\left(f_x \cdot x_s + f_y \cdot y_s\right) + s\left(f_x \cdot x_t + f_y \cdot y_t\right)$$

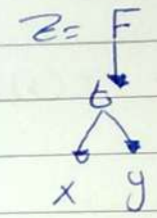
$$= t\left(f_x \cdot 2s + f_y \cdot -2s\right) + s\left(f_x \cdot -2t + f_y \cdot 2t\right)$$

$$= 2st\, f_x - 2st\, f_y - 2st\, f_x + 2st\, f_y$$

$$= \boxed{0}$$

Example : let $z = f(x^2 - y^2)$ prove that $y z_x + x z_y = 0$

Pf : $z = f(t)$, $t = x^2 - y^2$



$$y z_x + x z_y = y f'(t) \cdot t_x + x f'(t)\, t_y$$

$$= y f'(t)\, 2x + x f'(-2y)$$

$$= 2xy f' - 2xy f'$$

$$= 0$$

The implicit function Thm : (IFT)

If the eq. $f(x, y, z) = 0$ defines implicity a func. z in ~~the~~ terms of x, y , then $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$

عكس

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

Example : Find $\frac{dy}{dx}$ if $\frac{x^3 + y^3}{6xy} = 1$

Using the IFT

Sol :

$$\frac{x^3+y^3}{6xy} = 1$$

$$x^3+y^3 = 6xy$$

$$\underbrace{x^3+y^3-6xy}_{f(x,y)} = 0 \Rightarrow$$ لازم يكون اليمين يساوي صفر

$$\frac{dy}{dx} = \frac{-Fx}{Fy} = -\frac{3x^2-6y}{3y^2-6x}$$

Example : find Zx , Zy using the IFT , where

$$\frac{x^3y^3+z^3-1}{1-6xyz} = 0$$

$$\frac{x^3y^3+z^3}{1-6xyz} = 1 \Rightarrow x^3y^3+z^3 = 1-6xyz$$

$$\Rightarrow \underbrace{x^3y^3+z^3-1+6xyz}_{F(x,y,z)=0} = 0$$

$$Zx = \frac{-fx}{fz} = \frac{-(3x^2y^3+6yz)}{3z^2+6xy}$$

$$Zy = \frac{-fy}{fz} = \frac{-(3y^2x^3+6xz)}{3z^2+6xy}$$

$$\frac{\partial x}{\partial y} = \frac{-Fy}{Fx} = \frac{-3y^2x^3+6xz}{3x^2y^3+6yz}$$

$$\frac{\partial x}{\partial x} = \frac{-fz}{fx} = \frac{-(3z^2+6xy)}{3x^2y^3+6yz}$$

Sec 14.6 The Directional Derivative and the Gradient vector

Def: the gradient vector of a func $f(x,y,z)$ is

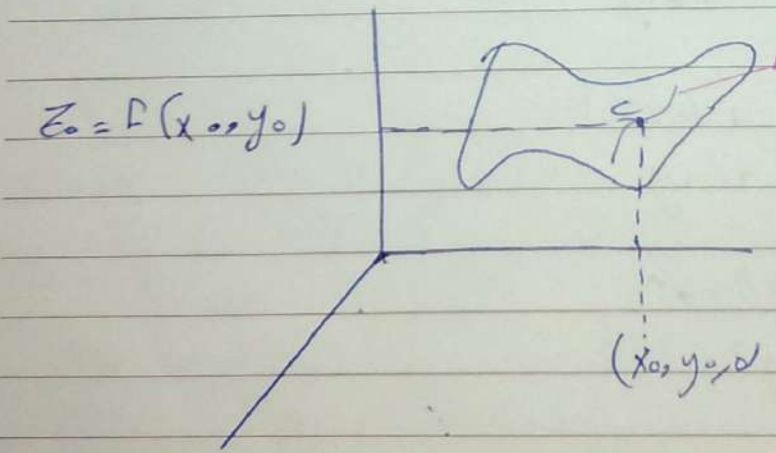
$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$
$$= \langle f_x i, f_y j, f_z k \rangle$$

* $f(x,y) \Rightarrow \nabla f = \langle f_x, f_y \rangle$

Def: the directional Derivative of the function $f(x,y)$ at a point (x_0, y_0, z_0) in the direction of a unit vector $\hat{u} = \langle a, b \rangle$ is

$$D_{\hat{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

* Geometric Interpolation (التفسير الهندسي)



Curve the intersection of the plane and the surface

plane P pass (x_0, y_0, z_0), $(x_0, y_0, 0)$ and parallel to $\hat{u}$

$D_u f(x_0, y_0) =$ slope of the tangent line to C (curve)
(where tangent lies in the plane P)

Thm : $D_{\hat{u}} f(x,y) = \nabla f \cdot \vec{u}$

Example : Find the directional derivative of $f(x,y,z) = x \sin(yz)$ at the pt. $A(1,3,0)$ in the direction of

$\vec{v} = \hat{i} + 2\hat{j} - \hat{k}$

Sol : $\nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle \sin yz, \; xz\cos(yz), \; xy\cos(yz) \rangle$

$\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$

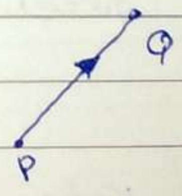
$|\vec{v}| = \sqrt{6} \neq 1 \Rightarrow \hat{v} = \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \rangle$

$D_{\hat{v}} f(1,3,0) = \langle 0,0,3 \rangle \cdot \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \rangle$

Remark : $D_{\hat{u}} f(x_0,y_0)$ = rate of change of $f(x,y)$ at the pt. (x_0,y_0) in the direction of $\hat{u}$.

Example : Find the rate of change of $f(x,y) = xe^y$ at the pt. $(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$

Sol : $\vec{u} = \overrightarrow{PQ} = \langle \frac{-3}{2}, 2 \rangle$



$|\vec{u}| = \sqrt{\frac{9}{4}+4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$

$\therefore \hat{u} = \langle \frac{-3}{5}, \frac{4}{5} \rangle$

$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$

$\nabla f(2,0) = \langle 1, 2 \rangle$

S_1: $z = x^2 + y^2$ and S_2: ...

rate of change $= D_{\hat{u}} f(2,0) = \nabla f(2,0) \cdot \hat{u}$

$= \frac{-3}{5} + \frac{8}{5} = \boxed{1}$

Remark: The max value of $D_{\hat{u}} f(x_0, y_0)$ (max rate of change) is in the direction of $\hat{u} = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$ is $|\nabla f(x_0, y_0)|$

The min value of $D_{\hat{u}} f(x_0, y_0)$ (min rate of change) is $-|\nabla f|$ and holds in the direction of $\hat{u} = \frac{-\nabla f}{|\nabla f|}$

Example: If $f(x,y) = x e^y$

1. Find the max rate of change of f at the pt $(2,0)$. In what direction does f has this max value

2. find the min rate of change of f at the pt. $(2,0)$ In what direction does f has this ~~max~~ min Value.

Sol: $\nabla f = \langle f_x, f_y \rangle = \langle e^y, x e^y \rangle$

$\nabla f(2,0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$

(1) max rate of change $=$ max $D_{\hat{u}} f(2,0) = |\nabla f(2,0)| = \sqrt{5}$

in the direction of $\hat{u} = \frac{\nabla f(2,0)}{|\nabla f(2,0)|} = \frac{\langle 1, 2 \rangle}{\sqrt{5}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

(2) min rate of change $=$ min $D_{\hat{u}} f(2,0) = -|\nabla f(2,0)|$

$= -\sqrt{5}$

in the direction of $\hat{u} = \frac{-\nabla f(2,0)}{|\nabla f(2,0)|} = \frac{-\langle 1, 2 \rangle}{\sqrt{5}} =$

$\langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

Example: find a unit vector $\hat{u}$ s.t $\nabla f(x_0, y_0) = 3i - 4j$

$D_{\hat{u}} f(x_0, y_0) = -5$.

Sol: $|\nabla f(x_0, y_0)| = \sqrt{25} = 5$

$D_{\hat{u}} f(x_0, y_0) = -5 = -|\nabla f(x_0, y_0)|$

$\Rightarrow \hat{u} = \dfrac{-\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|} = \dfrac{-3i - 4j}{5} = \dfrac{-3i}{5} + \dfrac{4}{5} j$

Example: let $\hat{u} = \frac{3}{5} i - \frac{4}{5} j$

$\hat{v} = \frac{4}{5} i + \frac{3}{5} j$

$D_{\hat{u}} f(1, 2) = -5$
$D_{\hat{v}} f(1, 2) = 10$

Find: ① $f_x(1, 2)$, $f_y(1, 2)$

② find the direction derivative of f at $(1, 2)$ in the direction that makes the angle $\theta = \frac{\pi}{6}$

Sol: $\nabla f(1,2) = \langle a, b \rangle$
$D_{\hat{u}} f(1,2) = -5 \Rightarrow \nabla f(1,2) \cdot \hat{u} = -5$
$\frac{3}{5} a - \frac{4b}{5} = -5$

$\boxed{3a - 4b = -25}$ ①

intersection of the traces of the surfaces S_1 and S_2 in the

$D_{\hat{v}} f(1,2) = 10 \Rightarrow \nabla f(1,2) \cdot \vec{v} = 10$

$\frac{4}{5}a + \frac{3}{5}b = 10$

$\boxed{4a + 3b = 50} \rightarrow \boxed{2}$

$3(1) + 4(2) \Rightarrow \quad 25a = 125 \quad \Rightarrow \boxed{a = 5}$

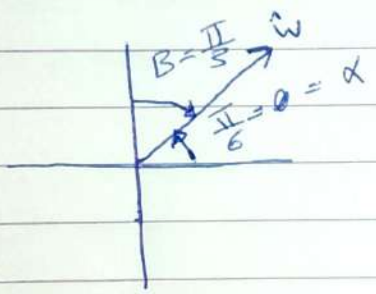
$3b = 30 \Rightarrow \boxed{b = 10}$

$\nabla f = \langle 5, 10 \rangle = \langle f_x(1,2), f_y(1,2) \rangle$

$\therefore f_x(1,2) = 5$

$\therefore f_y(1,2) = 10$

2



فرع ٢ ما بنحل الا اذا طلع فرع واحد.

$\hat{w}$ direction

$\Rightarrow$ direction angles of $\hat{w}$ are $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$,

$\hat{w} = \langle \cos\alpha, \cos\beta \rangle$

$= \langle \cos\frac{\pi}{6}, \cos\frac{\pi}{3} \rangle$

$= \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$D_{\hat{w}} f(1,2) = \nabla f(1,2) \cdot \hat{w}$

$= \langle 5, 10 \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$= \frac{5\sqrt{3}}{2} + \frac{10}{2}$

Remark 8:

$D_{\hat{i}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)$

$D_{\hat{j}} f(x_0, y_0, z_0) = f_y(x_0, y_0, z_0)$

$D_{\hat{k}} f(x_0, y_0, z_0) = f_z(x_0, y_0, z_0)$

Example

$D_{\hat{i}} f(1,2) = -5$

$D_{\hat{j}} f(1,2) = 10$

$\boxed{1}$ find $f_x(1,2)$, $f_y(1,2)$

(2) find the Direction divative of f at $(1,2)$ in the direction that makes the angle $\theta = \frac{\pi}{6}$

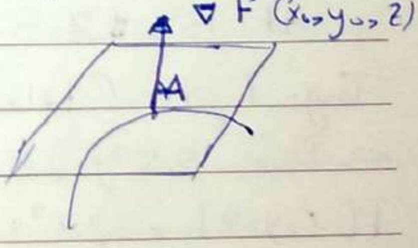
Sol: $D_{\hat{i}} f(1,2) = -5 \Rightarrow f_x(1,2) = -5$

$D_{\hat{j}} f(1,2) = 10 \Rightarrow f_y(1,2) = 10$

سؤال سهل
ع الكل ربح بس
كده في معطاه لا تكتب مرة =

Rule: let S is the surface $F(x_0, y, z) = 0$ at $A(x_0, y_0, z_0)$ is a pt on S

(1) $\nabla f(x_0, y_0, z_0)$ is a normal to the tangent plane to S At A



eq. of tangent plane is

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\nabla f_{(x_0, y_0, z_0)} = \langle a, b, c \rangle$

(2) A line L is a normal to the tangent plane to S at A

$\Leftrightarrow$ param. eqs. of L:

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

Example : find the eqs. of the tangent plane and param. eq. of the normal line to the surface $\frac{x^2}{4} + y^2 = 3 - \frac{z^2}{9}$

at the pt. $(-2, 1, -3)$

Sol : $f(x, y, z) = \frac{x^2}{4} + y^2 - 3 + \frac{z^2}{9}$

$\nabla f = \langle \frac{2x}{4}, 2y, \frac{2z}{9} \rangle$

$\nabla f(-2, 1, -3) = \langle -1, 2, \frac{-2}{3} \rangle$

tangent plane : $-1(x - -2) + 2(y-1) + \frac{-2}{3}(z - -3) = 0$

param. eqs. Normal line : $x = -2 - t$

$y = 1 + 2t$

$z = -3 - \frac{2}{3} t$

Example : find the param. eqs. of the line through the pt. $A(1,1,1)$ and parallel to the normal line of the surface $z = 2x^2y + 3xy^2$ at the pt. $B(1,1)$

at B : $z = 2(1)^2(1) + 3(1)(1)^2 = 5$ نوجد النقطة

Tangent pt. $(1, 1, 5)$ #

surface $2x^2y + 3xy^2 - z = 0$

$f(x, y, z) = 2x^2y + 3xy^2 - z$

$\nabla f = \langle \cancel{4xy, 6xy}\ 4xy + 3y^2, 2x^2 + 6xy, -1 \rangle$

$\nabla f(1, 1, 5) = \langle 7, 2, -1 \rangle$

Param. eq. $x = 1 + 7t$

$y = 1 + 2t$

$z = 1 - t$

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Sec 14.7 : Maximum and Minimum Values.

Def :

A function $f(x,y)$ at a pt $A(x_0, y_0, z_0) \in Dom(f)$ is said to have

(1) a local max (min) at A If $f(x,y) \leq f(x_0, y_0)$ ($f(x,y) \geq f(x_0,y_0)$) for all pts. (x_0, y_0) in some disk in Dom (f) with center at (x_0, y_0). The number $f(x_0, y_0)$ is called the local max. (min) of f at A.

(2) A local extrema if $f(x,y)$ has a local max. or local min at A.

(3) absolute max. (min) at A if $f(x_0, y_0) \geq f(x,y)$

$(f(x_0, y_0) \leq f(x,y))$ for all $(x,y) \in Dom(f)$

the max. (min) value of f is $f(x_0, y_0)$

(4) Absolute extrema at if f has an absolute max. or absolute min at A

Def : A pt. $(x_0, y_0) \in Dom\, f(x,y)$ is called a critical pt. of f If $f_x(x_0, y_0) = 0$, and $f_y(x_0, y_0) = 0$

or $f_x(x_0, y_0)$ ~~does~~ DNE , or $f_y(x_0, y_0)$ DNE

Example : Find a, b s.t $f(x,y) = x^2y + 3axy^2 - bxy$ has a critical pt. at $(1,-1)$

Sol :
$$f_x = 2xy + 3ay^2 - by$$
$$f_y = x^2 + 6axy - bx$$

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$f_x(1,-1) = 0 \Rightarrow -2 + 3a + b = 0$

$3a + b = 2 \rightarrow ①$

$f_y(1,-1) = 0 \Rightarrow 1 - 6a - b = 0$ * طالما [illegible] ما الـ ح عوضنا

$-6a - b = -1 \rightarrow ②$

$① + ② \Rightarrow -3a = 1 \Rightarrow a = \frac{-1}{3}$

$1 + b = 2 \Rightarrow \boxed{b = 1}$

2^{nd} Derivative test

suppose that the ~~secon~~ 2^{nd} Derivatives of $f(x,y)$ are Cont s. on a disk centered at a pt (a,b) and let $f_x(a,b) = 0$, $f_y(a,b) = 0$

let $D = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$

1. $D > 0$, $f_{xx}(a,b) > 0 \Rightarrow f$ has a local min (L. min.) at (a,b)
   $f(a,b)$ L. min. value

2. $D > 0$, $f_{xx}(a,b) < 0 \Rightarrow f$ has a local max. (L. max) at (a,b)
   $f(a,b)$ L. max. value

3. $D < 0 \Rightarrow f$ has a saddle pt. at (a,b) [f has neither a local max. ~~o~~ nor a L. min. at (a,b)]

Examples: classify the critical pts of 1. $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$, as L. max., L. min. or Saddle pt.

Sol 8) $f_x = 6x^2 + 6y^2 - 150 = 0$

$6x^2 + 6y^2 = 150 \quad \div 6$

$x^2 + y^2 = 25 \rightarrow ①$

$f_y = 12xy - 9y^2 = 0$

$3y(4x - 3y) = 0$

$3y = 0 \quad , \quad 4x - 3y = 0$

$y = 0 \quad , \quad y = \frac{4}{3}x$

If $y = 0 \Rightarrow ① : x^2 = 25$

$x = \pm 5$

$\boxed{(\pm 5, 0) \text{ critical pts}}$

If $y = \frac{4}{3}x \Rightarrow ①$

$x^2 + \frac{16}{9}x^2 = 25$

$\frac{25}{9}x^2 = 25$

$x^2 = 9$

$x = \pm 3$

$x = 3 \Rightarrow y = \frac{4}{3}(3) = 4$

$x = -3 \Rightarrow y = \frac{4}{3}(-3) = -4$

$\boxed{(3, 4), (-3, -4) \text{ critical pts}}$

$f_{xx} = 12x$

$f_{yy} = 12x - 18y$

$f_{xy} = 12y$

$D = (12x)(12x - 18y) - [12y]^2$

$\boxed{2}$ $f(x,y) = x^2 + y^2 - 2x - 6y + 2$

$f_x = 2x - 2 = 0 \Rightarrow x = 1$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

f has only one Critical pt $(1, 3)$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 0$

$D = 2(2) - 0^2 = 4$

$D(1,3) = 4 > 0$

$f_{xx}(1,3) = 2 > 0$

$\therefore$ f has a local min at $(1,3)$

pt	D	f_{xx}	Type of pt.
$(5,0)$	$60(60) \Rightarrow +$	$12(5) = +$	local min $\Rightarrow$ L.min $= f(5,0)$
$(-5,0)$	$-60(-60) \Rightarrow +$	$12(-5) = -$	local max. at $(-5,0)$
$(3,4)$	$12(3)(12(3)-18(4)) - (12(4))^2$ $\boxed{-}$	+	Saddle pt. at $(3,4)$
$(-3,-4)$	$12(-3)(12(-3)-18(-4)) - (12(-4))^2$ $\boxed{-}$	*	Saddle pt. at $(-3,-4)$

3) $f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3 \rightarrow ①$

$f_y = 4y^3 - 4x = 0 \Rightarrow y^3 = x \rightarrow ②$

①, ②

$x = (x^3)^3$
$x = x^9$

$\Rightarrow x = 0,\ x = -1,\ x = 1$

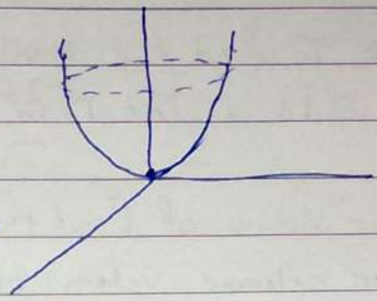
$y = 0,\ y = -1,\ y = 1$

f has 3 critical pts

$(0,0), (-1,-1), (1,1)$

Example: $f(x,y) = x^2 + y^2$

$z = x^2 + y^2$



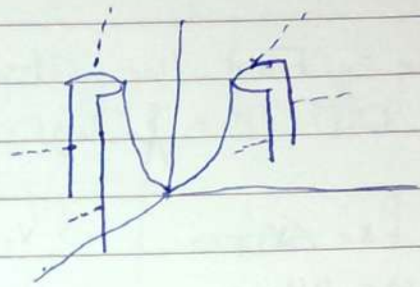
Sol: f has ~~not~~ at (0,0) an absolute min

$f(0,0) = 0$ is the absolute min value

F has no absolute max.

Example: $f(x,y) = y^2 - x^2$

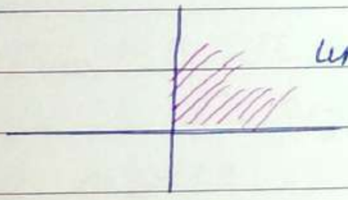
$z = y^2 - x^2$

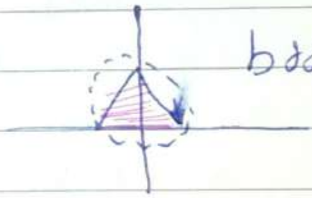


Sol:

F has no absolute extrema

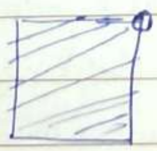
Remark: A region D in $\mathbb{R}^2$ is bounded (bdd) if D lies inside some circle





2) A region D in $\mathbb{R}^2$ is closed if its boundary pts. belongs to D

closed, bdd



not closed, bdd

not closed, bdd

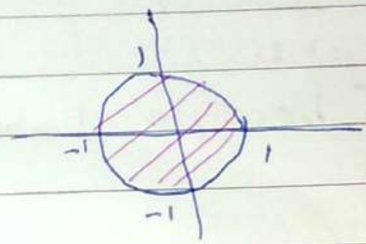
Extreme Value Thm for functions in 2 Variables:

If $f(x,y)$ conts on a closed Bdd set D in $\mathbb{R}^2$, then f has absolute max, and absolute min at pts in D

Remark: To find the absolute extrema of conts function $f(x,y)$ on a closed Bdd, let D in $\mathbb{R}^2$:

1. Find the values of f at the critical pts. inside D
2. Find the extreme values of f on the boundry of D
3. the largest value of f in (1)&(2) is the absolute max of f
4. the smallest value of f in (1)&(2) is the absolute min of f

Example: Find the absolute extrema of $f(x,y) = 2x^3 + y^4$ on the Disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$



Sol: $f_x = 6x = 0 \Rightarrow x = 0$

$f_y = 4y = 0 \Rightarrow y = 0$

$(0,0) \in D$

$\notin D$ لو نقطة

لا نأخذها $\in$

Step 2 on the Boundry of D

حدود يعوض عن الدائرة $\Leftarrow$ $x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2$

$$g(x) = f\Big|_{y^2 = 1-x^2} = 2x^3 + (1-x^2)^2$$

$g(x) = 2x^3 + 1 - 2x^2 + x^4 \quad x \in (-1,1) \Rightarrow$ ضمن المجال

$g'(x) = 6x^2 - 4x + 4x^3 = 0$ — نقاط الحرجة عدا أطراف الفترة

$= 2x(3x - 2 + 2x^2) = 0$ — ٢. كمشتقة

$2x = 0 \quad , \quad 2x^2 + 3x - 2 = 0$

$x = 0 \quad\quad (2x-1)(x+2) = 0$

$x = 0 \quad , \quad x = \frac{1}{2} \quad , \quad x = -2$

Cancle $\Rightarrow$ خارج المجال

Critical pts $x = 0, \frac{1}{2}, -1, 1$

$x = 0 \Rightarrow y^2 = 1 - x^2 \Rightarrow y = \pm 1$

$x = \frac{1}{2} \Rightarrow y^2 = 1 - x^2 \Rightarrow y = \pm \frac{\sqrt{3}}{2}$

$x = -1 \Rightarrow y^2 = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y^2 = 0 \Rightarrow y = 0$

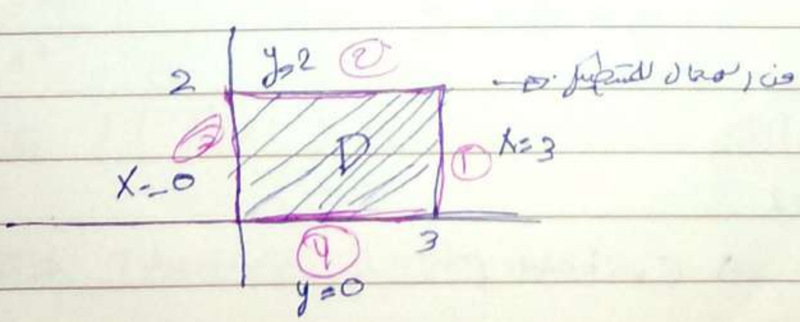
* critical pts. $(1,0), (-1,0), (0,1), (0,-1), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, \frac{-\sqrt{3}}{2})$
$(0,0)$

pt	$(0,0)$	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$(\frac{1}{2}, \frac{-\sqrt{3}}{2})$
F	0	2	-2	1	1	$\frac{13}{16}$	$\frac{13}{16}$

absolute max. of f is 2 holds at $(1,0)$
absolute min of f is -2 holds at $(-1,0)$

Example 8 Find the absolute extrema of $f(x,y) = x^2 - 2xy + 2y$, on the ~~interv~~ rectangle $D = \{(x,y) : 0 \le x \le 3, \ 0 \le y \le 2\}$

Sol 8:



Step 1

$f_x = 2x - 2y = 0 \Rightarrow y = x$ (1)

$f_y = -2x + 2 = 0 \Rightarrow x = 1$ (2)

$\Rightarrow y = 1$ by (1)

$(1,1) \in D \Rightarrow$ Critical Point (ضمن المجال)

Step 2

(1) $x = 3$

$g_1(y) = f(3, y) = 9 - 6y + 2y = 9 - 4y$, $\underbrace{0 \le y \le 2}$

$g_1'(y) = -4 \neq 0$

Critical pts. $y = 0 \Rightarrow x = 3$

$y = 2 \Rightarrow x = 3$

$(3,0)$, $(3,2) \rightarrow$ Critical pts

(2) $y = 2$

$g_{(2)}(x) = f(x, 2) = x^2 - 4x + 4$, $0 \le x \le 3$

$g_2' = 2x - 4 = 0$ $\boxed{x = 2}$

$x = 0$, $x = 2$, $x = 3$

$y = 2$, $y = 2$, $y = 2$

$(0,2)$, $(2,2)$, $(3,2) \Rightarrow$ Critical pts.

(3) $x = 0$

$g_3(y) = 2y$ $0 \le y \le 2$

$g_3' = 2 \neq 0$

$y = 0$, $y = 2$

$(0,0)$, $(0,2) \Rightarrow$ Critical pts.

(4) $y = 0$

$g_4(x) = x^2$ $0 \le x \le 3$

$g_4'(x) = 2x = 0$ $x = 0$

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Second #

$(0,0)$, $(3,0)$ critical pts.

pt.	$(1,1)$	$(3,0)$	$(3,2)$	$(0,2)$	$(2,2)$	$(0,0)$
f	1	9	1	4	0	0

$\therefore$ absolute max is 9 holds at $(3,0)$

$\therefore$ absolute min is 0 holds at $(2,2)$ & $(0,0)$

* الاشتقاق بالنسبة لـ local مشتقته في D بعطيه الـ absolute مشتقته في الـ critical pt

ch. 15 Multiple Integrals

Sec 15.1 Double integrals

$\iint f(x,y)\,dx\,dy$ or $\iint f(x,y)\,dy\,dx$ * السؤال لحد معين تكامل

Example : $\iint y^2 e^{xy}\,dx\,dy = \int y^2 \frac{e^{xy}}{y}\,dy = \int y e^{xy}\,dy$

$= \int y e^{xy}\,dy$

$$\begin{array}{cc} y & e^{xy} \\ 1 & \frac{e^{xy}}{x} \\ 0 & \frac{e^{xy}}{x^2} \end{array} \qquad = \frac{y}{x} e^{xy} - \frac{e^{xy}}{x^2} + c$$

Remark : $\iint f(x)\,g(y)\,dx\,dy = \left(\int f(x)\,dx\right)\left(\int g(y)\,dy\right)$

15.2 Iterated Integrals:-

Thm : let $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

$$\iint f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,\underline{\underline{dy}}\,dx$$

المفاضلة

← نعمله أول حاجة ونعوض الحدود عشان $\underline{\underline{y}}$

Five Apple

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

$\partial A \to \partial y\, \partial x$

$\partial A \to \partial x\, \partial y$

انت الي بتختار

Rule ⇒ (1) $\iint_{\boxed{R}} (f + y)\, \partial A = \iint_R f\, \partial A + \iint_R y\, \partial A$

Iterated ← حدود

* بعد تبديل التكاملات بشرط أبدل حدود التكامل ← يشترط تكون التكاملات من حدود لعدد ← إذا كان حدود التكامل حسن أعداد ما بصير تبدل بين dy و dx ← صحيحة

(2) $\iint_R c f\, \partial A = c \iint_R f\, \partial A$

(3) $f(x,y) \geq g(x,y)$ on a region R in $\mathbb{R}^2$

$$\iint_R f\, \partial A \geq \iint_R g\, \partial A$$

* تبديل التكاملات كما هي يكون صحيح اذا كانت حدود التكامل جميعها أعداد

Example:

$\boxed{1}$ $\iint (x - 3y^2)\, dy\, \partial x = \int (xy - y^3)\, \partial x = \frac{y x^2}{2} - y^3 x$

$\boxed{2}$ $\int_0^{\frac{\pi}{2}} \int_0^{\pi} \sin x \cos y\, dy\, \partial x$ iterated

$$= \left(\int_0^{\frac{\pi}{2}} \sin x\, \partial x\right) \left(\int_0^{\pi} \cos y\, \partial y\right)$$

$$-\cos x \Big]_0^{\frac{\pi}{2}} * \sin y \Big]_0^{\pi}$$

$$1 * 0 = \boxed{0}$$

Example: Find $\iint_R y \sin(xy)\, dA$, Where $R = \{ (x, y) : 1 \le x \le 2,\ 0 \le y \le \pi \} = [1, 2] \times [0, \pi]$

Sol: $\iint_R y \sin(xy)\, dA = \int_0^{\pi}\int_1^{2} y \sin(xy)\, dx\, dy$

$$= \int_0^{\pi} \frac{y(-\cos(xy))}{y} \Big]_1^2 \, dy$$

$$= \int_0^{\pi} \cos(2y) - \cos y \,]\, dy$$

$$= -\left[\frac{\sin(2y)}{2} - \sin y \right]_0^{\pi} = \boxed{0}$$

Ex: $I = \iint_D \sqrt{1-x^2}\, dA$. where $dA = [-1, 1] \times [2, 4]$

Sol: $I = \int_{-1}^{1}\int_{2}^{4} \sqrt{1-x^2}\, dy\, dx$

let $y = \sqrt{1-x^2}$

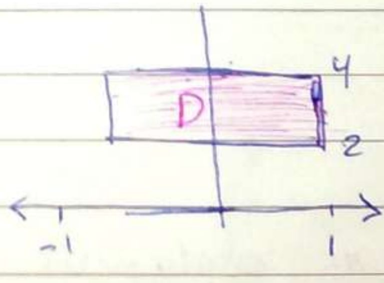
$y^2 = 1 - x^2$

$y^2 + x^2 = 1$

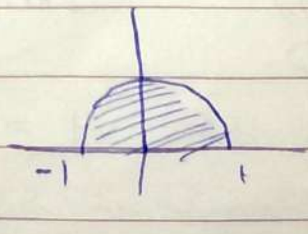
$$= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

$$= 2 \cdot \frac{1}{2} \pi \cdot (1)^2$$

$$= \pi$$



* اعتمدنا على مساحة الدائرة

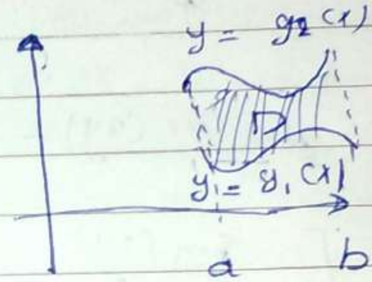


Sec 15.3 Double integrals over general Regions

Type 1 Region

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

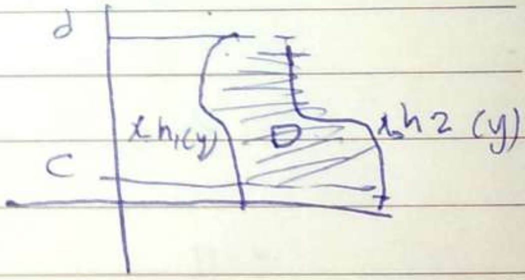
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f \,dy\,dx$$



Type 2 Region

$$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f \,dx\,dy$$



Ex 8) evaluate

(1) $I_1 = \iint_D (x+2y)\,dA$, where D is the region

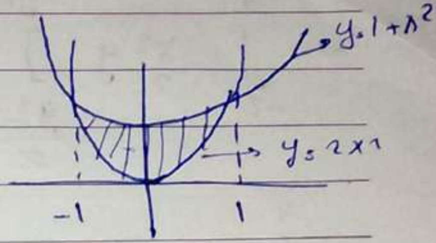
enclosed By $y = 2x^2$, $y = 1 + x^2$

[illegible]

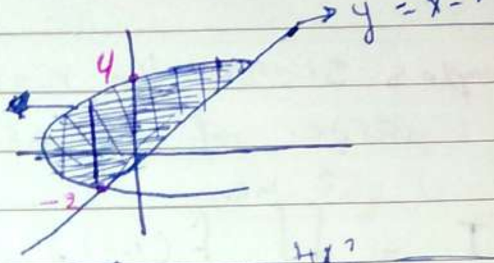
$2x^2 = 1 + \cancel{x}^2$ نقاط تقاطع

$x^2 = 1 \qquad x = \pm 1$

$$I_1 = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$



$\boxed{2}$ $I_2 = \iint_R xy\, dA$, where $\mathbb{R}$ is the shaded region in the figure



Type 2 Region

$x = \dfrac{y^2 - 6}{2}$

$x = y + 1$

$\dfrac{y^2 - 6}{2} = y + 1$

$y^2 - 6 = 2y + 2$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = 4 \ , \ y = -2$

بوضع الحدود بناءً على نقطة
من جهة العرض.

$$I_2 = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy\, dx\, dy$$

$$I_1 = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} xy + y^2 \Big]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} x(1+x^2) + (1-x^2)^2 - x\,2x^2 - (2x^2)^2 \Big] dx$$

$$= \int_{-1}^{1} x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4 \Big] dx$$

$$= \int_{-1}^{1} [x - x^3 + 1 + 2x^2 - 3x^4] dx$$

$= \ldots$

$$= \int_{-2}^{4} \frac{x^2}{2} y \Big]_{\frac{y^2-6}{2}}^{y+1} dy$$

$$= \frac{1}{2}\int_{-2}^{4} y(y+1)^2 - \left(\frac{y^2-6}{2}\right)^2 y \; dy$$

$$= \frac{1}{2}\int_{-2}^{4} \left[y^3+2y^2+y - \left(\frac{y^5-12y^3+36y}{4}\right)\right] dy$$

Example: Sketch the region of integration and change the order of integration

(1) $I_1 = \int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx$

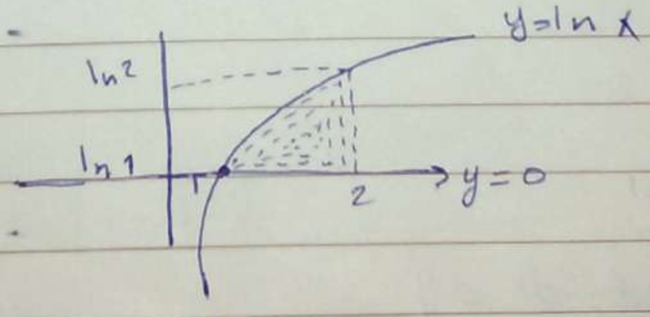
(2) $I_2 = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{2} f(x,y)\, dx\, dy$

(3) $I_3 \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} f(x,y)\, dx\, dy$

Sol: (1) D: $y=0 \longrightarrow y=\ln x$

$1 \le x \le 2$

Sketch $\longrightarrow$



change $\longrightarrow$

$dy\,dx \longrightarrow dx\,dy$

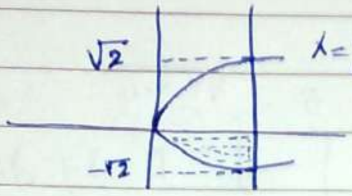
Type 1 $\qquad$ Type 2 $\Rightarrow$ $x = e^y$ $\qquad 0 \le y \le \ln x$

$x = 2$

$$I_1 = \int_0^{\ln 2}\int_{e^y}^{2} f \, dx \, dy$$

$\boxed{2}$ D : أيسر $x = y^2 \longrightarrow$ أيمن $x = 2$

$-\sqrt{2} \le y \le 0$



$y^2 = 2$, $y = \pm\sqrt{2}$ نق تقاطع

Type 2 $\longrightarrow$ Type 1

$dx\,dy$ $\quad$ $dy\,dx$

$y = \pm\sqrt{x}$

لأني اكون بتو منطقتها السالبة

$$I_2 = \int_0^{2}\int_{-\sqrt{x}}^{0} f \, dy \, dx$$

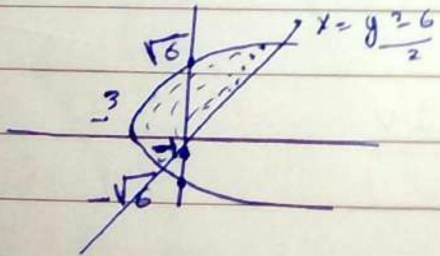
$y = -\sqrt{x}$

$y = 0$

$0 \le x \le 2$

$\boxed{3}$ Region $x = \frac{y^2 - 6}{2} \longrightarrow x = y + 1$

$-2 \le y \le 4$



$x = \frac{y^2-6}{2} \Rightarrow$ كرتنا مفتوحة لليمين هنا y^2 موجب ... تكون y values بالعكس

Type 2 $\quad$ Type 1

$dx\,dy$ $\quad$ $dy\,dx$

$y = -\sqrt{2x+6} \longrightarrow y = \sqrt{2x+6}$

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$$-\sqrt{2x+6} = x-1$$

$$2x+6 = x^2-2x+1$$

$$x^2-4x-5=0$$

$$(x-5)(x+1)=0$$

$x=5$ $\quad x=-1$

$$I_3 = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f\, dy\, dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f\, dy\, dx$$

Example:

فصل المتغيرات Evaluate: (1) $I_1 = \int_0^{1/2}\int_{2x}^{1} \sin y^2\, dy\, dx$

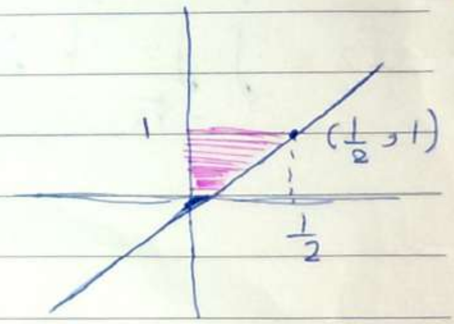
(2) $I_2 = \int_0^2\int_{y/2}^{1} e^{x^2}\, dx\, dy$

Sol: $\boxed{1}$

$dy\,dx \rightarrow dx\,dy$

$0 \le x \le \frac{1}{2}$, $y=2x$, $y=1$

$x=\frac{y}{2}$, $x=0$, $0\le y\le 1$



$$I_1 = \int_0^1\int_0^{y/2} \sin y^2\, dx\, dy$$

$$= \int_0^1 \left[x \sin y^2\right]_0^{y/2} dy$$

$w=y^2 \rightarrow dw = 2y\,dy$, $dy = \frac{dw}{2y}$

$$\frac{1}{4}\int_0^1 \sin w\, dw = \frac{1}{4}\left(-\cos w\right)\Big]_0^1$$

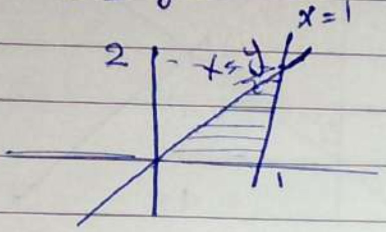
$$= \frac{-1}{4}\left[\cos 1 - 1\right]$$

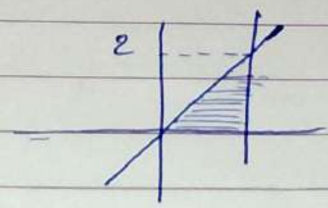
$$= \frac{1}{4} - \frac{1}{4}\cos 1$$

$\boxed{2}$ $dx\,dy \longrightarrow dy\,dx$

$x = \frac{y}{2} \longrightarrow y = 2x$

$0 \le y \le 2$





$$I_2 = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 2x\, e^{x^2}\, dx$$

$w = x^2$

$dw = 2x\, dx$

$\frac{dw}{2x} = dx$

$$I_2 = \int_0^1 \cancel{2x} e^{w} \frac{dw}{\cancel{2x}}$$

$x = 0 \Rightarrow w = 0^2 = 0$

$x = 1 \Rightarrow w = 1^2 = 1$

$$= e^{w} \Big]_0^1$$

$$= e^1 - e^0 = \boxed{e - 1}$$

سؤال مهم

Example: Combine the sum of the 2 double integrals as a single ~~to~~ double integral.

$$I = \int_0^{\frac{1}{2}} \int_0^{2y} f\, dx\, dy + \int_{\frac{1}{2}}^{1} \int_0^{2-2y} f\, dx\, dy$$

D_1

$dx\,dy$

$x = 0 \Rightarrow x = 2y$

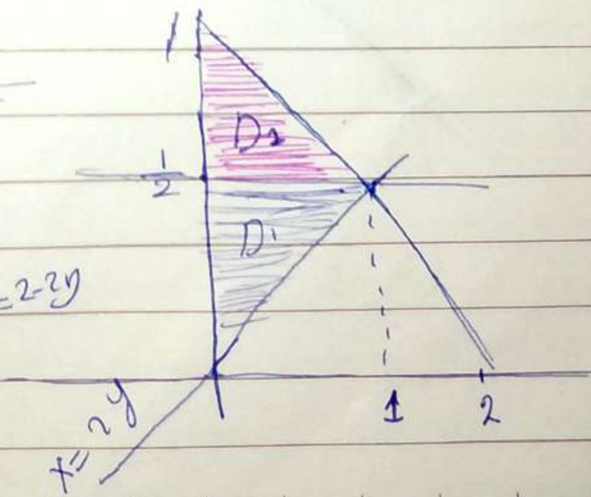
$0 \le y \le \frac{1}{2}$

D_2

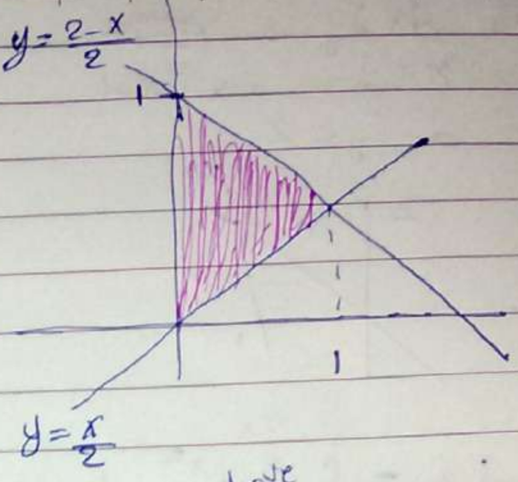
$dx\,dy$

$x = 0 \Rightarrow x = 2 - 2y$

$\frac{1}{2} \le y \le 1$



$$I = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} f \, dy \, dx$$



* Rules: The volume of the solid bdd above by the surface $z_1 = f_1(x,y)$, bdd below by the surface $z_2 = f_2(x,y)$ and the projection of the solid on the xy plane is the region D is

$$V = \iint_R (f_1 - f_2) \, dA$$

Example: find the volume of the solid lies under $z = x^2 + y^2$ and lies above the region D in the xy-plane bdd by: $y = 2x$, $y = x^2$

Sol: Surfaces $z = x^2 + y^2$ [illegible]

$z = 0$ [illegible] ⇒ من خلال الشكل يقع فوق

[illegible] ⇐ xy-plane

D:

$$V = \iint_D (x^2 + y^2 - 0) \, dA$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \, dy \, dx$$

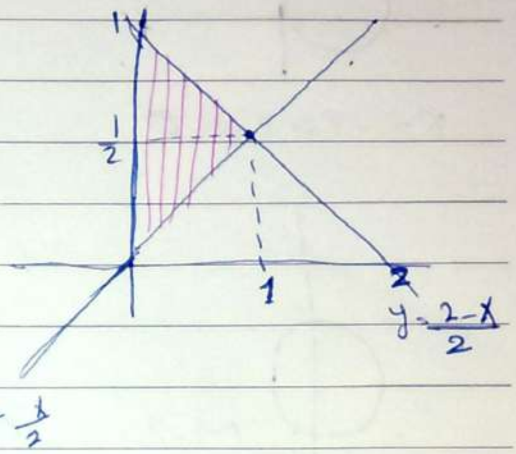
Example: set up as a double integral but do not evaluate The Volume tetrahedron. bdd by $x+2y+z=2$, $x=0$, $z=0$, $x=2y$

Sol: Surfaces $z=2-x-2y$
$z=0$

D:- $x=0$

$x=2y$, $y=\frac{x}{2}$

$2-x-2y=0$ $y=\frac{2-x}{2}$

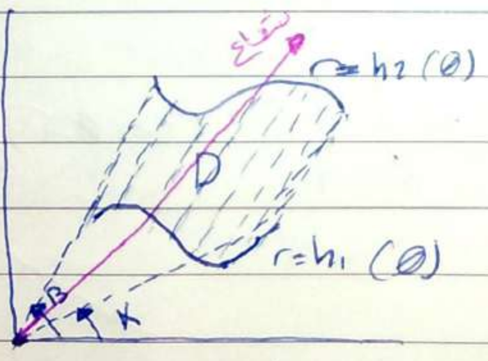


$$V=\int_0^1\int_{\frac{x}{2}}^{\frac{2-x}{2}}(2-x-2y-0)\,dy\,dx$$

حدناسين، نفرض مين x محور من
خلاه نقطة بالطريقة، مخلاص (حش)
نعوض المعطى بتاخذ عضو صغير
كنقطة معينة (0,0) أو (0,1)

Sec 15.4

D: region in the x-y plane as in the figure.

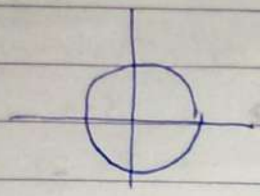


اذا كان، نقطة معينة
لل origin

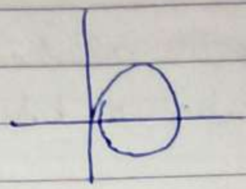
$\alpha < B$, $0 \le B-\alpha \le 2\pi$

$$\iint_D f(x,y)\,dA=\int_{\alpha}^{B}\int_{h_1\theta}^{h_2\theta} f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}$$

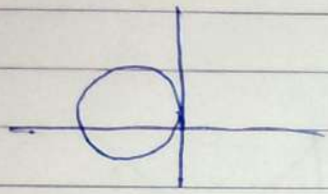
الترتيب ضرب



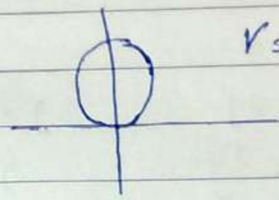
$r = a$



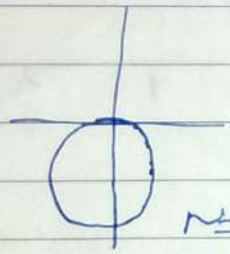
$r = 2a\cos\theta$



$r = -2a\cos\theta$



$r = 2a\sin\theta$



$r = -2a\sin\theta$ where $a > 0$

Example: Evaluate $I = \iint_R (3x + 4y^2)\, dA$

where R is the region in the upper half plane bdd by

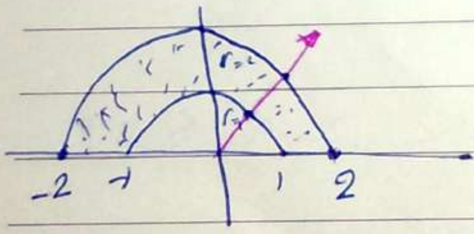
دوائر كلها [illegible]

$x^2 + y^2 = 1$

$x^2 + y^2 = 4$

$$I = \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

Sol:

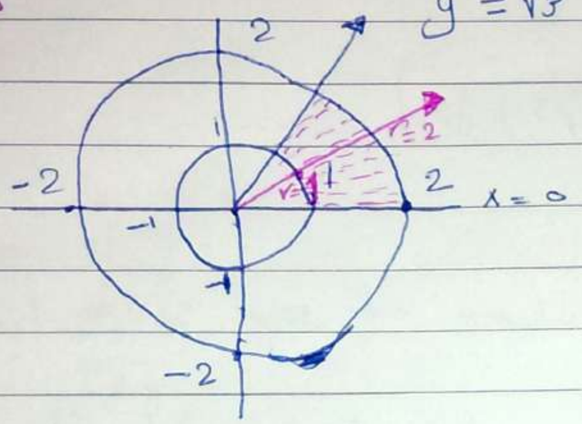


Example: Evalute $I = \iint_D \tan^{-1}\frac{y}{x}\, \partial A$

Where $D = \{(x,y) : 1 \le x^2+y^2 \le 4 \;,\; 0 < y \le \sqrt{3}\,x\}$

Sol: $y = \sqrt{3}\,x$

المنطقة جزء من دائرة ← بولر



$$y = \sqrt{3}\,x \implies \tan\theta = \frac{y}{x} = \sqrt{3}$$

$$I = \iint_D \tan^{-1}\frac{y}{x}\,\partial A = \int_0^{\pi/3}\int_1^{2} \theta\, r\, dr\, d\theta$$

$$= \left(\int_0^{\frac{\pi}{3}} \theta\, d\theta\right)\left(\int_1^2 r\, dr\right) = \ldots\ldots$$

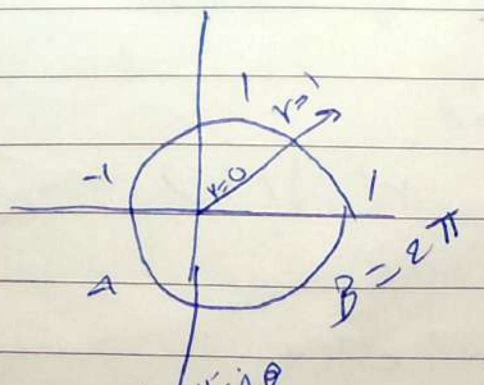
Example: find the Volume of the Solid bdd by $z=1$ and $z = 2 - x^2 - y^2$

Sol: Surface $z = 1$

$z = 2 - x^2 - y^2$

$1 = 2 - x^2 - y^2$

$$\boxed{x^2 + y^2 = 1}$$



$$r = \iint_D \underbrace{2 - x^2 - y^2}_{\text{العلوي}} \; - \underbrace{1}_{\text{السفلي}} \,) \, \partial A$$

بعرف العلوي والسفلي من خلال آخر نقطة داخل الدائرة.

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r \,\partial r \,\partial \theta$$

تكاملات منفصلة لأنها أرقام

$$= \left(\int_0^{2\pi} \partial\theta \right) \left(\int_0^1 (r - r^3)\, \partial r \right) = \text{-----}$$

Example : find the volume of the solid lies under

$z = \sqrt{x^2+y^2}$ above the xy- plane and inside

$x^2 + y^2 = 2x$

Sol :

$z = \sqrt{x^2+y^2}$
$z = 0$
$\Big\} \to$ $x^2 + y^2 = 0$ معادلة نقطة من سطح

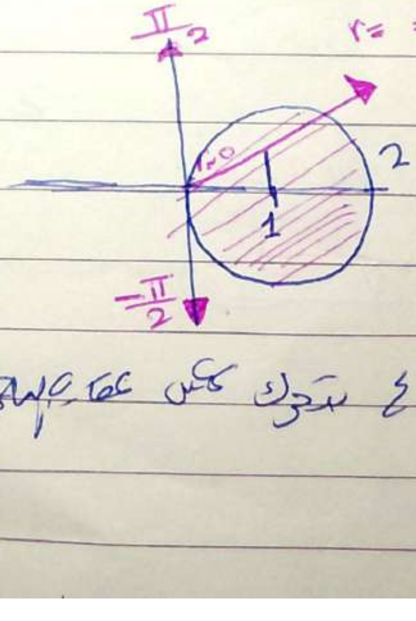
D : $x^2 + y^2 = 2x$ $\Rightarrow$ مركزه 1
قطره 2

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$

المقطع

$r = \underline{\underline{2}}\cos\theta$



$$V = \iint_D \sqrt{x^2+y^2} - 0 \,)\, \partial A$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \,\partial r \,\partial \theta$$

الشعاع يتحرك عكس عقارب الساعة

* حاصل الضرب هون

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$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\cos^3\theta}{3}\, d\theta = \ldots$$

Example: Evaluate

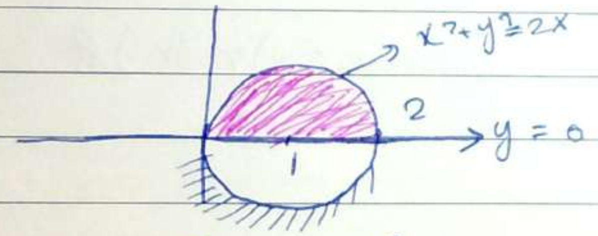
$\boxed{1}$ $$I_1 = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$\boxed{2}$ $$I_2 = \int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\, dy\, dx$$

$\boxed{3}$ $$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\, dx\, dy.$$

Sol: $\boxed{1}$ $y = 0 \longrightarrow y = \sqrt{2x - x^2}$

$0 \le x \le 2$ $\quad y^2 = 2x - x^2 \Rightarrow \boxed{x^2 + y^2 = 2x}$

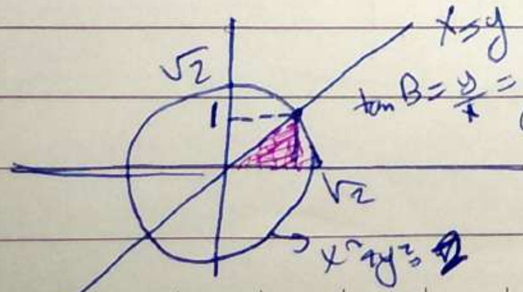


$$I_1 = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\; r\, dr\, d\theta = \ldots\ldots$$

* تربيع تحت جذر و معامل
التربيع ⇐ دائرة.

$\boxed{3}$ $x = y$ $\quad x = \sqrt{2-y^2}$

$0 \le y \le 1$ $\quad x^2 = 2 - y^2 \Rightarrow x^2 + y^2 = 2$



$\tan\theta = \frac{y}{x} = 1 \quad \theta = 45^\circ$

نقط تقاطع المنحنيين

$y^2 + y^2 = 2$

$y = \pm 1$

$$I_3 = \int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\, \partial r\, \partial\theta$$

Example : Combine the sum a single double integral

طلع اول اشي

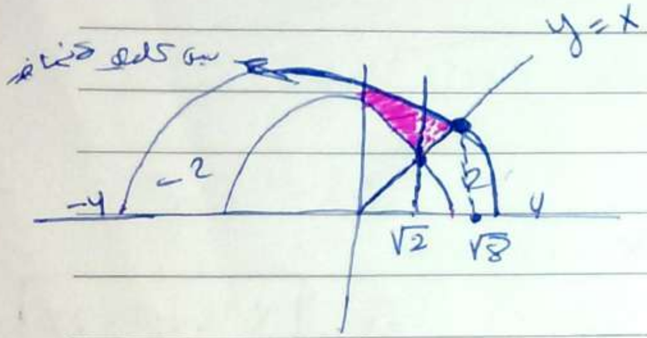
$$I = \int_0^{\sqrt{2}} \int_{\sqrt{4-x^2}}^{\sqrt{16-x^2}} f(x,y)\, \partial y\, \partial x + \int_{\sqrt{2}}^{\sqrt{8}} \int_{x}^{\sqrt{16-x^2}} f(x,y)\, dy\, dx .$$

Sol : $y = \sqrt{4-x^2} \rightarrow y = \sqrt{16-x^2}$ (1) (2)

$0 \le x \le \sqrt{2}$

(3) (4) $y = x \rightarrow y = \sqrt{16-x^2}$

$\sqrt{2} \le x \le \sqrt{8}$



$x^2 + y^2 = 4$

$y = x$

$\Rightarrow x^2 + x^2 = 4$

$2x^2 = 4$

$x = \sqrt{2}$

$y^2 + x^2 = 16$

$y = x$

$\Rightarrow x^2 + x^2 = 16$

$2x^2 = 16$

$x = \sqrt{8}$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_2^4 f(r\cos\theta, r\sin\theta)\, r\, \partial r\, \partial\theta$$

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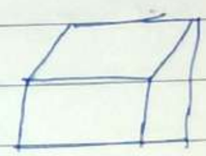
Sec 15.7 Triple Integrals :

$B = [a,b] \times [c,d] \times [r,s] = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

$$\iiint_B f(x,y,z)\, dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\, dz\, dy\, dx$$

$$= \int_c^d \int_r^s \int_a^b f \quad dx\, dz\, dy$$ ← [ترتيبات.

Ex : $B = \underbrace{[0,1]}_{x} \times \underbrace{[2,3]}_{y} \times \underbrace{[-1,5]}_{z}$ ← B



$$\iiint_B x y^2 z^3\, dV =$$

Sol : $$\int_{-1}^{5} \int_{2}^{3} \int_{0}^{1} x y^2 z^3\, dx\, dy\, dz$$

$$= \left(\int_0^1 x\, dx\right) \left(\int_2^3 y^2\, dy\right) \left(\int_{-1}^{5} z^3\, dz\right)$$

\# **Rule :** let S be the solid bdd by

1. $g_1(x,y) \le z \le g_2(x,y)$, D the projection of S on the xy-plane $\Rightarrow \iiint_S f(x,y,z)\, dV = \iint_D \left[\int_{g_1}^{g_2} f\, dz\right] dA_{xy}$

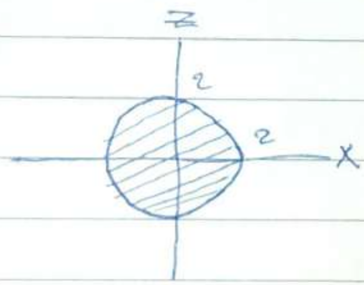
(2) $h_1(x,z) \le y \le h_2(x,z)$, D the projection of S' on the xz-plane $\Rightarrow \iiint_S f\,\partial v = \iint_D \left[\int f\,\partial y\right] \partial A_{xz}$

(3) $u_1(y,z) \le x \le u_2(y,z)$, D the Projection of S' on the yz plane $\Rightarrow \iiint_S f\,\partial v = \iint_D \left[f\,\partial x \right] \partial A_{yz}$

Example: Evalute $I = \iiint_E \sqrt{x^2+z^2}\,\partial v$, where E is the Solid Bdd by $y = x^2+z^2$, $y=4$

Sol: surface $y = x^2+z^2$
$y=4$

D_{xz}: $x^2+z^2=4$



$$I = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,\partial y \right] \partial A$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^{4} \overbrace{r\,\partial y\,r}^{r^2}\,\partial r\,\partial \theta$$

Example: Express ~~iterated~~ Iterated integral $I = \int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,\partial z\,\partial y\,\partial x$ in a different order:

(1) first integrate with respect to x, then y, then z.

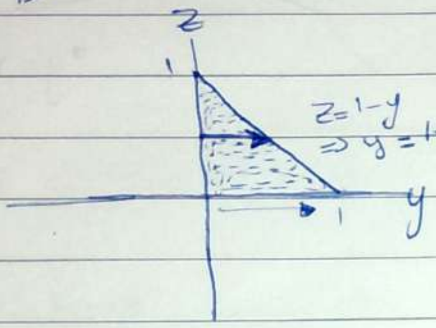
(2) First integrate with respect to y, then x, then z.

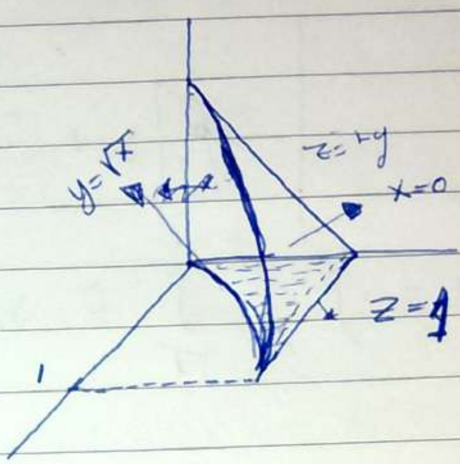
Sol 8) surfaces $z = 0 \rightarrow z = 1-y$

D : $y = \sqrt{x} \rightarrow y = 1$

$0 \le x \le 1$

$\boxed{1}$ $I = \iint_D \int_0^{y^2} f \, dx \, dy \, dz$ → type 2

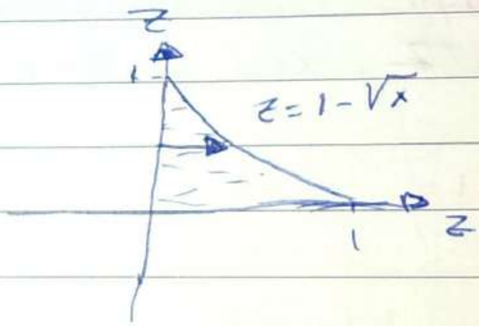




$$I = \int_0^1 \int_0^{1-z} \int_0^{y^2} f \, dx \, dy \, dz$$

$\boxed{2}$ $\iint_{D_{xz}} \int_{\sqrt{x}}^{1-z} f \, dy \, dx \, dz$

$$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f \, dy \, dx \, dz$$



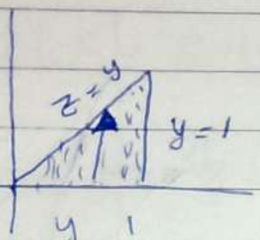
Example: Express the ~~integ~~ iterated integral $I = \int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z) \, dz \, dy \, dx$ as iterated integral with order of integration with x, then z, then y

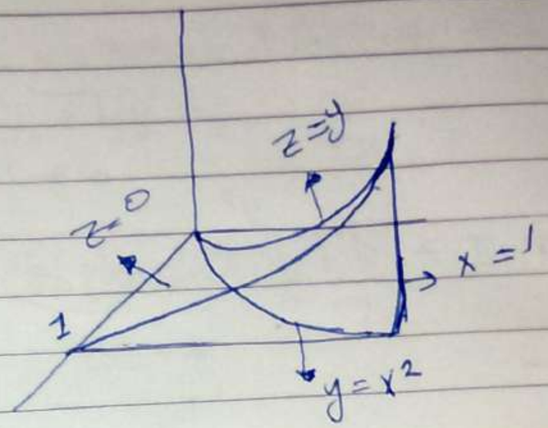
surfaces : $z = 0 \rightarrow z = y$

D_{xy} : $y = 0 \rightarrow y = x^2$

$0 \le x \le 1$

$$I = \iint_D \int_{\sqrt{y}}^{1} f \, \partial x \, \partial z \, \partial y$$





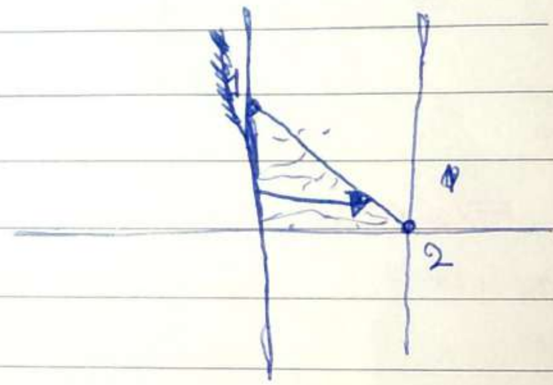
$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^{1} f \, \partial x \, \partial z \, \partial y$$

Thm: the Volume of solid S is $V = \iiint_S 1 \, \partial v$

Example: write the volume of the Solid bdd by $x + 2y + z = 2$, $x = z$, $x = 0$, $z = 0$, as an iterated triple integral

Sol: Surfaces: $z = 2 - x - 2y$, $z = 0$ $\Rightarrow$ $2 - x - 2y = 0$, $\boxed{x + 2y = 2}$, $\boxed{x = z}$, $\boxed{x = 0}$

$$V = \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 1 \, \partial z \, \partial x \, \partial y$$



Example: If $\iiint_G 5 \, \partial v = 13.5$

Find Volume of G?

$$\text{volume} = \frac{13.5}{5} \Longrightarrow$$ لانو لازم الحجم يكون واحد

Rule: Area of region R is $A = \iint_R 1 \, dA$

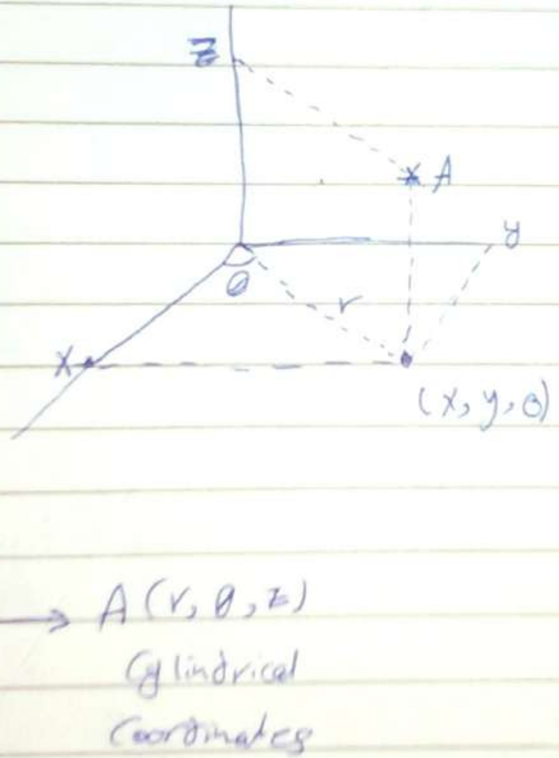
~~Int~~

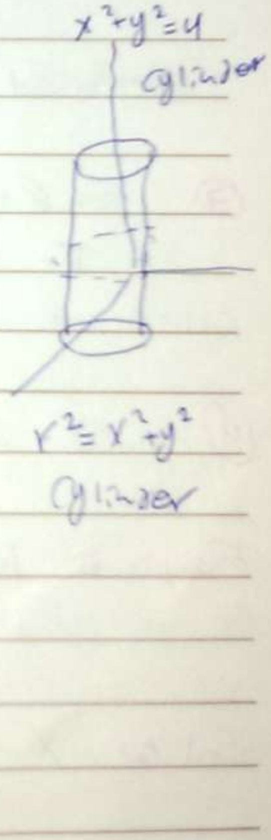
Example: If $\iint_D -2 \, dA = -13.5$, Find Area of D

Sol: Area $= \frac{-13.5}{-2}$

Sec 15.8: Triple integrals in cylindrical Coordinates

The cylindrical coordinates of the pt. $A(x,y,z)$ are $A(r, \theta, z)$
where $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2} \Leftrightarrow r^2 = x^2+y^2$
$0 \le \theta \le 2\pi$, $\tan\theta = \frac{y}{x}$



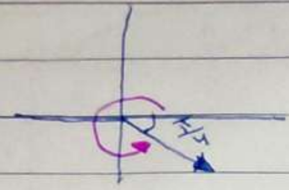


$A(x, y, z) \longrightarrow A(r, \theta, z)$

Rectangular Coordinates — Cylindrical Coordinates

Ex: Find the cylindrical coordinates of the pt A with rectangular coordinates ① $A(3, -3, -7)$ ② $A(-3, 3, -7)$ ③ $A(-3, -3, -7)$
④ $A(3, 3, -7)$

Sol :



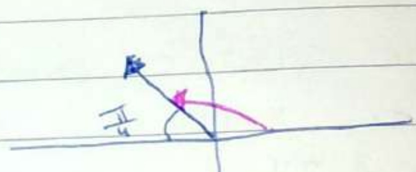
* لفيت آخرها مع عقارب الساعة
عشان عقارب الساعة [illegible] سالب

$\tan\theta = \frac{-3}{3} = -1$

$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$r = \sqrt{x^2+y^2} = \sqrt{18}$ $\Rightarrow$ cylindrical Coordinates $A(\sqrt{18}, \frac{7\pi}{4}, -7)$

② $r = \sqrt{18}$

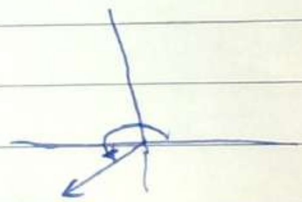


$\tan\theta = \frac{3}{-3} = \boxed{-1}$

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Cylindrical Coordinates $A(\sqrt{18}, \frac{3\pi}{4}, -7)$

③ $\tan\theta = \frac{-3}{-3} = 1$



$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

cylindrical coordinate $= (\sqrt{18}, \frac{5\pi}{4}, -7)$

④ cylind. coor. $(\sqrt{18}, \frac{\pi}{4}, -7)$

Example : $A(5, \frac{2\pi}{3}, 2)$ in cylind. Coord. then find the rectangular coord. of A

Sol : $x = 5\cos\left(\frac{2\pi}{3}\right) = 5\left(\frac{-1}{2}\right) = \frac{-5}{2}$

$y = 5\sin\left(\frac{2\pi}{3}\right) = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$

rectang. coord. $A\left(\frac{-5}{2}, \frac{5\sqrt{3}}{2}, 2\right)$

Ex: Convert the surface $z^2 = 3x^2 + y^2 + x$ to cylind. coor.

($y^2 + 2x^2$)

Sol → $z^2 = r^2 + 2r^2\cos^2\theta + r\cos\theta$

Exap: Convert to rectangular coord. $z = r^2\cos\theta - \sin\theta$

Sol: $zr = r^3\cos\theta - r\sin\theta$

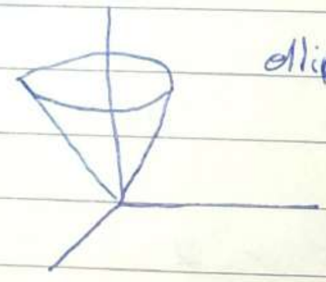
$z\sqrt{x^2+y^2} = (x^2+y^2)x - y$

$$\iiint_S f(x,y,z)\, dV = \iint_D \left(\int_{g_1}^{g_2} f(x,y,z)\, dz \right) dA$$

$$= \iint_{D_{r\theta}} \int_{g_1}^{g_2} f(r\cos\theta, r\sin\theta, z)\, dz \; r\, dr\, d\theta$$

Examples: Describe and Sketch the surface $z = r$

Sol: $z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2$

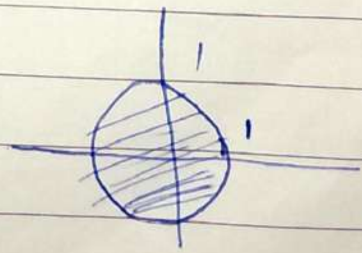


elliptic cone.

Examples: Find the Volume of the Solid within the cylinder $x^2+y^2=1$ below the plane $z=4$, and above the paraboloid $z = 1 - x^2 - y^2$

Sol: surface $z=4$

$z = 1 - x^2 - y^2$



$$V = \int_0^{2\pi}\int_0^{1}\int_{1-r^2}^{4} 1 \, \partial z \; r \, \partial r \, \partial \theta$$

Using cylindrical Coordinate ⇒ triple int. ثم نستخدم 8

Example: Evaluate :

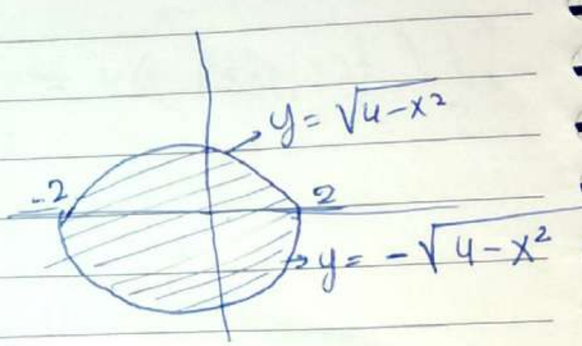
$$\boxed{1}\; I_1 = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2) \, \partial z \, \partial y \, \partial x$$

Sol: D : $y = -\sqrt{4-x^2} \rightarrow y = \sqrt{4-x^2}$

$-2 \leq x \leq 2$

$$I_1 = \int_0^{2\pi}\int_0^{2}\int_r^{z \to 2} r^2 \, \partial z \; r \, \partial r \, d\theta$$

$$I_1 = \int_0^{2\pi}\int_0^{2} r^3(2-r) \, \partial r \, \partial \theta \; \ldots\ldots$$

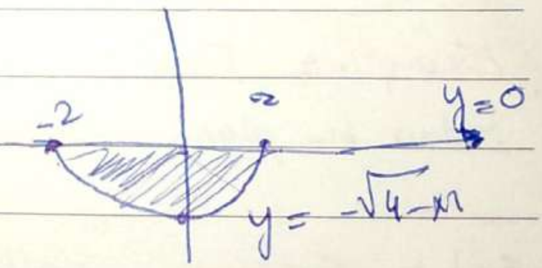


$$\boxed{2}\; I_2 = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{0}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2) \, \partial z \, \partial y \, \partial x$$

Sol: D : $y = -\sqrt{4-x^2} \rightarrow y = 0$

$-2 \leq x \leq 2$

$$I_2 = \int_{\pi}^{2\pi}\int_0^{2}\int_r^{2} r^2 \, \partial z \; r \, \partial r \, \partial \theta$$



✱ دايماً لما يكون عندك جذر زي كذا

تريح وحلها سهل عن طريق (cylind. coord.)

Example: Evalute $\Rightarrow I = \iiint_E (x+y+z)\, dV$, where E is the solid in the first octant that lies under the paraboloid $z = 12 - 3x^2 - 3y^2$

$y=0 \quad z=0 \quad x=0$

$y \geq 0 \quad z \geq 0 \quad x \geq 0 \rightarrow$ كلهم موجبة

Sol:

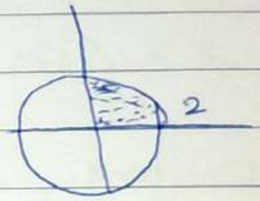
Surface: $z = 12 - 3x^2 - 3y^2$

$z = 0$

D: $x = 0$, $y = 0 \Rightarrow$ إقترانين بالمستوى

$12 - 3x^2 - 3y^2 = 0 \Rightarrow x^2 + y^2 = 4$

$$I = \int_0^{\pi/2} \int_0^2 \int_0^{12-3r^2} (r\cos\theta + r\sin\theta + z)\, dz\, r\, dr\, d\theta$$



Sec 15.9 Triple integrals in ~~Plane~~ Spherical coordinates:

let $A(x, y, z)$ be in rectangular coordinates

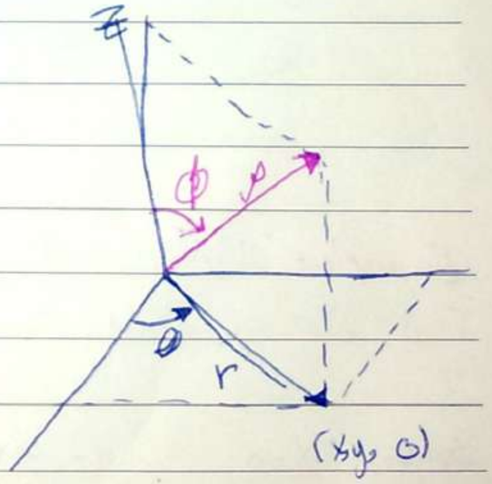
$\rho = \sqrt{x^2 + y^2 + z^2}$

$\rho^2 = x^2 + y^2 + z^2$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$



$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

The spherical coord. of A are $A(\rho, \theta, \phi)$

~~[illegible]~~ $r = \rho \sin\phi$

التدريس

Five Apple

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Example: Convert the pt. $A(2, \frac{\pi}{4}, \frac{2\pi}{3})$ to rectangular and cylindrical Coord. A is given in spherical coord.

Sol: $\rho = 2$, $\theta = \pi/4$, $\Phi = \frac{2\pi}{3}$

$x = \rho \sin\Phi \cos\theta$

$= 2 \sin\left(\frac{2\pi}{3}\right) \cos\frac{\pi}{4} = \frac{2\sqrt{3}}{2} \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}$

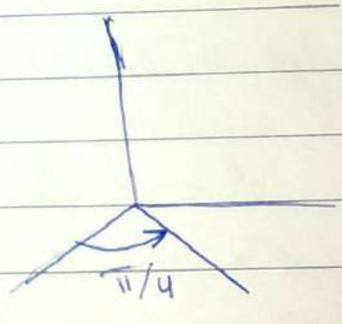
$y = \rho \sin\Phi \sin\theta = \frac{\sqrt{3}}{\sqrt{2}}$

$z = \rho \cos\Phi = 2\cos\frac{2\pi}{3} = 2\left(\frac{-1}{2}\right) = -1$

Rectang. Coord. of A are $A\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, -1\right)$

$r = \rho \sin\Phi = 2\sin\frac{2\pi}{3} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

Cylindr. Coord. are $A(\sqrt{3}, \frac{\pi}{4}, -1)$

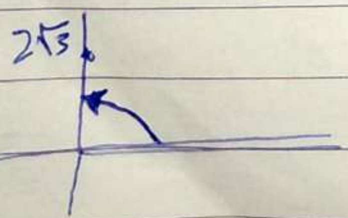


Example: The pt. $A(0, 2\sqrt{3}, -2)$ is in rectangular coordinates. Find the spherical Coord. of A.

Sol: $x = 0$ $y = 2\sqrt{3}$ $z = -2$

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{16} = 4$

θ:



$\theta = \pi/2$

$Z = \rho \cos\Phi \Rightarrow \cos\Phi = \frac{z}{\rho} \Rightarrow \cos\Phi = \frac{-2}{4} \Rightarrow \cos\Phi = \frac{-1}{2}$

$\Phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore A\ (4, \pi/2, \frac{2\pi}{3})$

Example: Convert the surface from spherical to rectangular Coordinates then sketch it :: $\boxed{1}$ $\Phi = \frac{\pi}{4}$ $\boxed{2}$ $\Phi = \frac{3\pi}{4}$ $\boxed{3}$ $\rho = 3$

Sol: $\boxed{1}$ $\cos\Phi = \cos\frac{\pi}{4}$

$\frac{z}{\rho} = \frac{1}{\sqrt{2}} \Rightarrow z = \frac{\rho}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sqrt{x^2+y^2+z^2}$

$z = \frac{1}{\sqrt{2}}\sqrt{x^2+y^2+z^2}$ regt. :: Cone

$z^2 = \frac{1}{2}(x^2+y^2+z^2)$

$2z^2 = x^2+y^2+z^2$

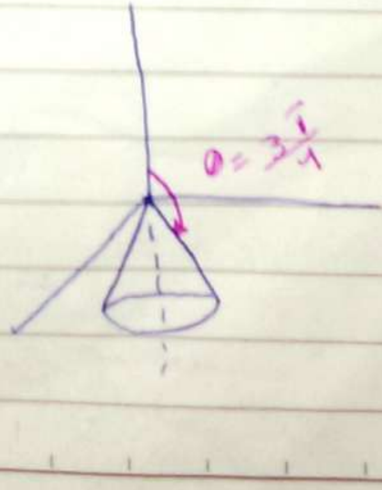
$z^2 = x^2+y^2$

الجذر

(2) $\cos\Phi = \cos\frac{3\pi}{4}$

$\frac{z}{\rho} = \frac{-1}{\sqrt{2}}$

$z = -\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{2}}$

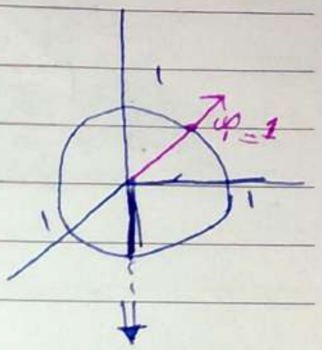


Five Apple

Example: Evaluate $I = \iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$

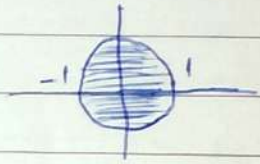
where $B : \{ (x, y, z) : x^2+y^2+z^2 \le 1 \}$

$$I = \int_0^{\pi} \int_0^{2\pi} \int_0^1 (e)^{(\rho^2)^{3/2}} \rho^2 \sin\Phi \, d\rho \, d\theta \, d\Phi$$



$$I = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 e^{\rho^3} \sin\phi \, d\rho \, d\theta \, d\Phi$$

$$= \left(\int_0^{\pi} \sin\Phi \, d\Phi \right) \left(\int_0^{2\pi} 1 \, d\theta \right) \cdot \left(\int_0^1 \rho^2 e^{\rho^3} \, d\rho \right)$$



Example: use spherical coord. to find the volume of the solid that

(1) above $z = \sqrt{x^2+y^2}$ and below (inside) $x^2+y^2+z^2 = z$

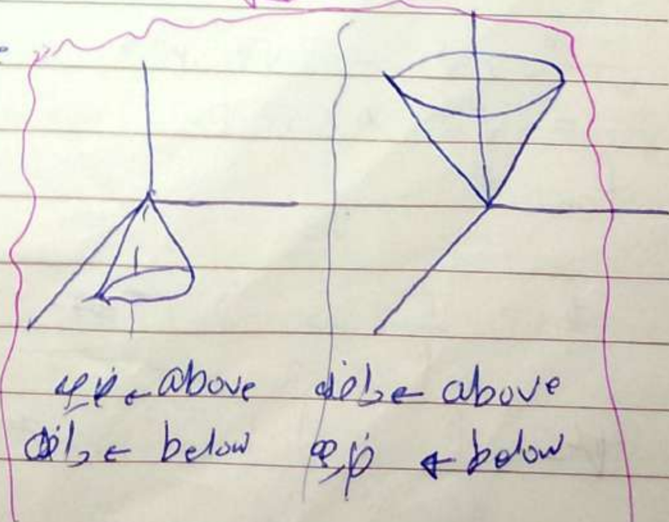
(2) inside $x^2+y^2+z^2 = 4$ and above the xy-plane, below $z = \sqrt{3x^2+3y^2}$

(3) inside $x^2+y^2+z^2 = 4$ and above $z = -\sqrt{\dfrac{x^2+y^2}{3}}$

Sol: [1] $x^2+y^2+z^2-z=0$

$x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ Sphere

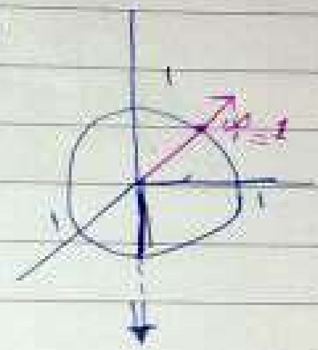
Remark



Example: Evaluate $I = \iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$

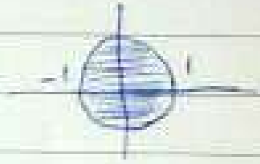
where $B: \{(x,y,z): x^2+y^2+z^2 \le 1\}$

$$I = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} (e)^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



$$I = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^2 e^{\rho^3} \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^{2\pi} 1 \, d\theta\right)\cdot\left(\int_0^{1} \rho^2 e^{\rho^3} \, d\rho\right)$$



Example: Use spherical coord. to find the volume of the solid that

(1) above $z = \sqrt{x^2+y^2}$ and below (inside) $x^2+y^2+z^2 = z$

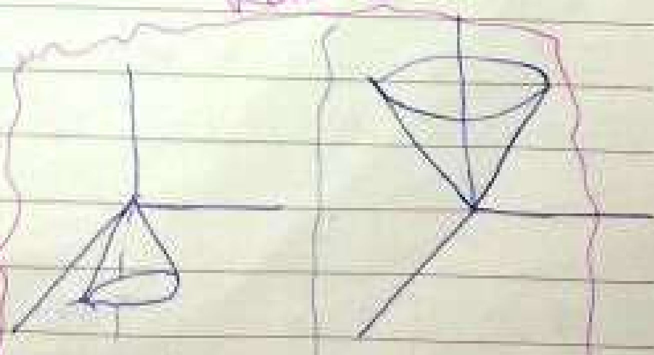
(2) inside $x^2+y^2+z^2=4$ and above the xy-plane, below $z = \sqrt{3x^2+3y^2}$

(3) inside $x^2+y^2+z^2=4$ and above $z = -\sqrt{\frac{x^2+y^2}{3}}$

Sol: (1) $x^2+y^2+z^2-z=0$

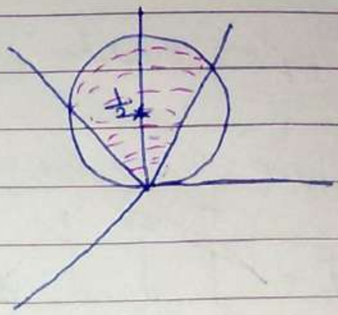
$x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ Sphere

Remark



كرة ← above   مخروط ← above

مخروط ← below   كرة ← below



$$V = \iiint_S 1 \, \partial V$$

$$= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\Phi} 1 \rho^2 \sin\Phi \, \partial\rho \, \partial\theta \, \partial\Phi$$

$x^2 + y^2 + z^2 = z$

$\rho^2 = \rho \cos\Phi$

$\rho = 0 \longrightarrow \rho = \cos\Phi$

$0 \leq \theta \leq 2\pi$

$\Phi = 0 \longrightarrow \Phi = \pi/4$

$z = \sqrt{x^2 + y^2} \Rightarrow$

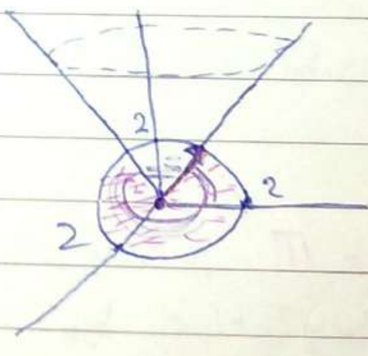
$\rho \cos\Phi = r \; \rho \sin\Phi$

$\tan\Phi = 1 \Rightarrow \Phi = \pi/4$

(2) $x^2 + y^2 + z^2 = 4$ sphere

$\rho = 0 \longrightarrow \rho = 2$

$0 \leq \theta \leq 2\pi$



$z = \sqrt{3} \sqrt{x^2 + y^2}$

$\rho \cos\Phi = \sqrt{3} \, r$

$= \sqrt{3} \, \rho \sin\Phi$

$\tan\Phi = \frac{1}{\sqrt{3}}$

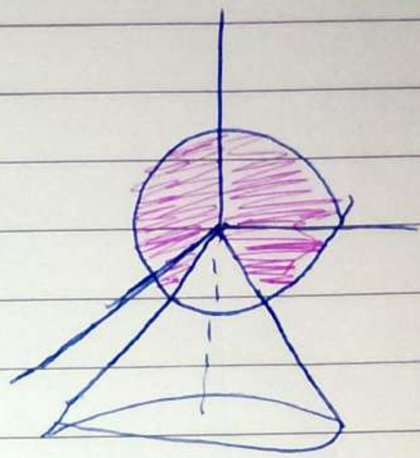
$\Phi = \frac{\pi}{6} \longrightarrow \Phi = \pi$

$$V = \int_{\pi/6}^{\pi} \int_0^{2\pi} \int_0^{2} 1 \cdot \rho^2 \sin\Phi \, \partial\rho \, \partial\theta \, \partial\Phi$$

(3) $\rho = 0 \rightarrow \rho = 2$

$0 \leq \theta \leq 2\pi$

$\Phi = 0 \rightarrow \Phi = \frac{2\pi}{3}$



$\rho \cos \Phi = \frac{-1}{\sqrt{3}} r = \frac{-1}{\sqrt{3}} \rho \sin \Phi$

$\tan \Phi = -\sqrt{3} \Rightarrow \frac{2\pi}{3} = \Phi$

$$\int_0^{\frac{2\pi}{3}} \int_0^{2\pi} \int_0^{2} 1 \cdot \rho^2 \sin \Phi \, d\rho \, d\theta \, d\Phi$$